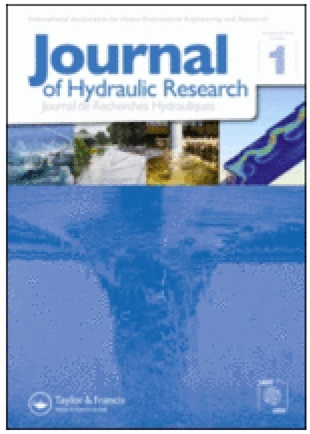

A TWO-FLUID MODEL FORMULATION OF THE BOUNDARY LAYER FLOW OF A VISCOUS INCOMPRESSIBLE FLUID PAST A FLAT PLATE AT ZERO INCIDENCE

Journal:	*Journal of Hydraulic Research*
Manuscript ID:	TJHR-2011-0198
Manuscript Type:	Paper
Date Submitted by the Author:	24-Oct-2011
Complete List of Authors:	Mhozya, Paul; Retired
Keywords:	boundary layer flow, primary flow, secondary flow, viscous incompressible fluid, energy balance, kinetic energy conservation

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PART I

Abstract

Due to apparent faults of the momentum operator in relation to rotational motion energy conservation has been employed to alternatively formulate the boundary layer flow problem. The boundary layer flow is modelled on two flows, the main flow parallel to flat plate surface and a secondary flow transverse to it. Then, the energy conservation equation $\Delta Q = \Delta E + \Delta W$ has been treated under the result to give simple partial differential equations easily solved by integration.

And on evaluation of a kronecker delta, the boundary layer flow is found to be governed by two equations, one relating to shear stresses and the other to normal stresses. These equations are not new because one of them is modelled on the Hagen-Poiseuille formula for the flow inside a circular pipe whereas the other corresponds to Bernoulli's law. Additional governing conditions include kinetic energy conservations, and when the flow is laminar continuity must also be satisfied.

Key words: Secondary flow, primary flow, boundary layer flow, viscous incompressible fluids, kinetic energy conservation, energy balance.

1.1 A brief comparative analysis

Secondary fluid flow formation is a common phenomenon associated with the drag effects of boundary layers on solid bodies immersed in fluid flows [2]. It is also known to occur when a fluid flows around bends in which case it is assumed to be caused by the centrifugal force arising from circulation. Whatever the cause, a secondary flow is a stream of fluid that flows from a flat solid surface that is immersed parallel to a main fluid stream. Such a property gives the secondary flow its own identity complete with its own component of velocity transverse to the flat surface. A boundary layer flow is then understood to be the region of the subsequent interaction between the main stream and the secondary stream as shown in figure 1.1.1. And due to the presence of viscous drag effects, the interaction between the two flows is effectively that of the main stream dragging the secondary flow streamlines along. This is essentially the basis of the two-fluid model description of the boundary layer flow where with secondary flow velocities u parallel to the surface and v transverse to it, the boundary layer flow then acquires the profile shown in figure 1.1.2.

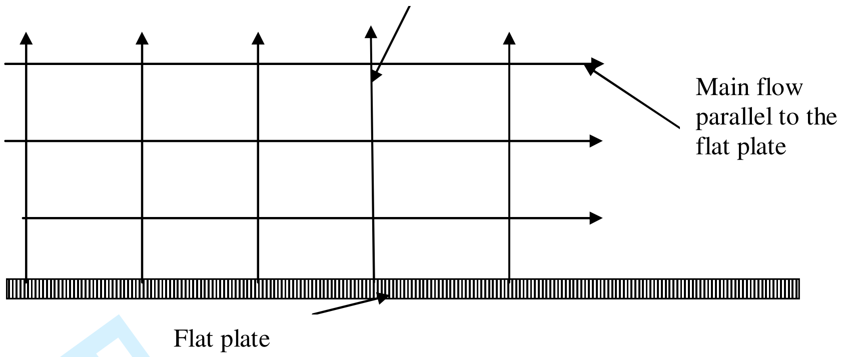

Figure 1.1.1: Ideal fluid secondary flow formation

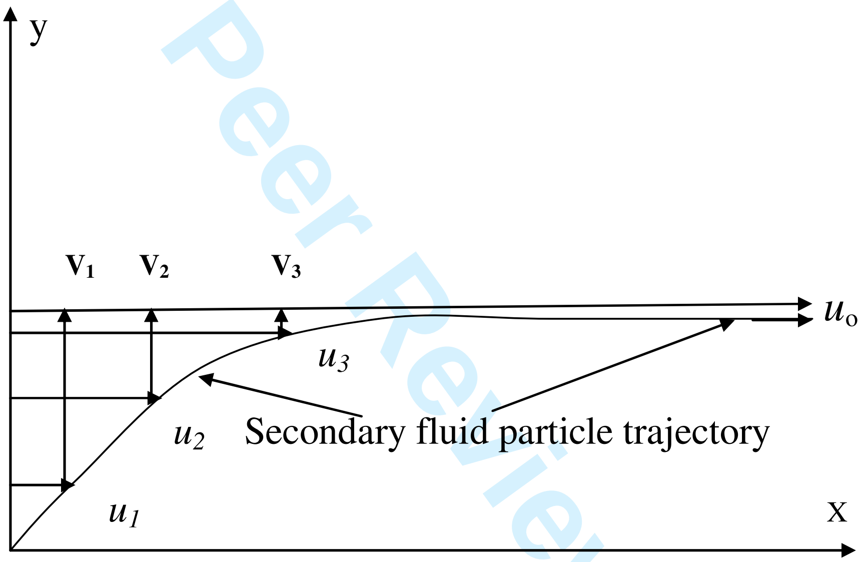

Figure 1.1.2: Actual secondary flow profile

The consequences of the two-fluid model profile in figure 1.1.2 above include the following basic observations:

- No-slip taken together with kinetic energy conservation $u^2 + v^2 \equiv u_o$ require that when $u = 0$ then $v = u_o$, so that v does not vanish on the plate surface.
- The maximum transverse velocity v_{max} is on the plate surface and decreases as the secondary stream becomes absorbed into the main stream.
- Viscous drag effects cause the velocity profiles for both the main flow and the secondary flow to exhibit changes in velocity perpendicular to their respective directions.

The three observations above portray two significantly negative features, that is, the first and second are obviously at variance with the Cauchy boundary conditions whereas the third observation is contrary the definition of momentum. All three observations are an obvious suggestion that the two-fluid model boundary layer flow problem above may not easily respond to the momentum formulation using the Cauchy-type boundary conditions. This possibility is further supported by the fact that the momentum operator ($\vec{v}\bullet\vec{\nabla}$) used to derive the N-S equations is inherently limited to the derivation and solution of rectilinear solid body motion only.

Consider a resultant force $(\vec{F_i})$ acting on a mass (m) that is moving with an instantaneous velocity $(\vec{V}(t))$ so that

$$\vec{F_i} = m\frac{d\vec{V}}{dt}$$

From it we obtain two different results, both of which exist in theory but one of which may not exist in practice. If we subject the RHS of the above force to partial differentiation in space and time we get the force

$$\vec{F}_1 = m\left\{\frac{\partial \vec{V}}{\partial t} + (\vec{V}\bullet\vec{\nabla})\vec{V}\right\} \qquad (1.1.1)$$

Alternatively we can postpone the partial differentiation and first calculate the work done

$$\vec{F_i}\bullet d\vec{r} = \frac{1}{2}m\,d\left(\vec{V}\bullet\vec{V}\right)$$

And if thereafter we apply the partial differentiation we have

$$\vec{F_2}\bullet\vec{V} = \vec{V}\bullet m\frac{\partial\vec{V}}{\partial t} + \vec{V}\bullet\frac{1}{2}m\vec{\nabla}(\vec{V}\bullet\vec{V})$$

from which we get the expression for the corresponding force

$$\vec{F_2} = m\frac{\partial\vec{V}}{\partial t} + \frac{1}{2}m\vec{\nabla}(\vec{V}\bullet\vec{V}) \qquad (1.1.2)$$

But vector algebra requires that for any two vectors $\vec{A}$ and $\vec{B}$

$$\vec{\nabla}\left(\vec{A}\bullet\vec{B}\right) = \left(\vec{B}\bullet\vec{\nabla}\right)\vec{A} + \left(\vec{A}\bullet\vec{\nabla}\right)\vec{B} + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)$$

where with $\vec{A}=\vec{V}$ and $\vec{B}=\vec{V}$ we have

$$\vec{\nabla}\left(\vec{V}\bullet\vec{V}\right) = 2\left(\vec{V}\bullet\vec{\nabla}\right)\vec{V} + 2\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right) \qquad (1.1.3)$$

From equation (1.1.3) it follows that

$$\frac{1}{2}\vec{\nabla}\left(\vec{V}\bullet\vec{V}\right) \neq \left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}$$

except when either $\vec{V}=0$ or $\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=0$. That is, $\vec{F}_1$ and $\vec{F}_2$ shall be equal if and only if either both do not exist or both are irrotational.

But the secondary flow streamline in figure 1.1.2 shows that a boundary layer flow is always rotational and $\vec{V} \times (\vec{\nabla} \times \vec{V}) \neq 0$ [1]. And since $\vec{F}_1$ lacks the rotational term, it is therefore incapable of formulating the trajectory of the secondary flow in figure 1.1.2 above.

Further, even though both of the quantities $\vec{\nabla}(\vec{V} \bullet \vec{V})$ and $(\vec{V} \bullet \vec{\nabla})\vec{V}$ have the dimensions of force, the former corresponds to a regular definition of a Newtonian force whereas the latter has no Newtonian physical equivalent. So $\vec{F}_1$ may actually not exist as a physical force and the only choice left open for us is adopt the force $\vec{F}_2$.

1.2 The Governing equations

Simple analysis shows that the force $\vec{F}_2$ from equation (1.1.2) cannot replace the force $\vec{F}_1$ from equation (1.1.1) directly in the formula for the Navier-Stokes equations. So as to abandon $\vec{F}_1$ in favor of $\vec{F}_2$, we must completely abandon the momentum formula and the corresponding Navier-Stokes equations. And since it has been shown above that $\vec{F}_1$ is actually contained in $\vec{F}_2$ [2], it is obviously any governing situation obtained using $\vec{F}_2$ must similarly contain some, if not all the ingredients of $\vec{F}_1$. Therefore, the best way to resolve the apparent contradictory behavior between the two forces is to find a common ground in which both forces share properties.

Accordingly, we note that except for the factor of $\frac{1}{2}$, both $\vec{F}_1$ and $\vec{F}_2$ have the same work-energy relations, that is

$$F_1 \bullet \vec{V} = \frac{1}{2} m \frac{\partial\, \vec{V} \bullet \vec{V}}{\partial\, t} + m\vec{V} \bullet \vec{\nabla}(\vec{V} \bullet \vec{V})$$

$$\vec{F_2} \bullet \vec{V} = \frac{1}{2} m \frac{\partial\, (\vec{V} \bullet \vec{V})}{\partial\, t} + \frac{1}{2} m\vec{V} \bullet \vec{\nabla}(\vec{V} \bullet \vec{V})$$

So basing our argument on this finding we can strongly assert that $\vec{F}_1$ can be successfully replaced by $\vec{F}_2$ by re-formulating our fluid flow problem in terms of energy. And one way of achieving this is to invoke the first law of thermodynamics and subsequently use energy balance [2][3] as our boundary layer flow governing condition.

In its basic form the energy conservation law states that the change in the total energy (E) is equal to the sum of the change in the heat energy (Q) and the work (W)

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[1] The secondary flow streamline approaching the free stream asymptotically testifies to this fact.

[2] And previously $\vec{F}_1$ has successfully been employed to solve a similar boundary layer problem.

done on the system. That is, using the symbol of change Δ, one alternative form of this law may be expressed as

$$\Delta E = \Delta Q - \Delta W \qquad 1.2\,(i)$$

The negative sign accompanying the work signifies the output work done by the system (as compared to an input work done on the system).

For an elementary mass (dm) of secondary fluid of density (ρ) having an elementary volume (dV), we have $dm = \rho\, dV$. And if a unit mass of the same fluid flowing with velocity $\left(\vec{V}\right)$ under a total head $\left(\vec{H}\right)$ possesses internal energy (de), translational kinetic energy $\left(\frac{1}{2}d\left(\vec{V}\bullet\vec{V}\right)\right)$ and gravitational potential $\left(\vec{g}\bullet\vec{H}\right)$, that elementary mass ($dm$) then possesses the total energy (dE) given by

$$dE = \rho\, dV\left(de + \frac{1}{2}d\left(\vec{V}\bullet\vec{V}\right) + \vec{g}\bullet\vec{H}\right) \qquad 1.2(ii)$$

The viscous drag between the secondary flow streamline and the main flow is the same as that between flat planes in relative motion. Under such viscous action, the two planes moving with relative velocity (∂u_i) in the x_i-direction experience two stresses, τ_{ii} in the x_i-direction and τ_{ij} in the x_j-direction. Taken with the coefficient of dynamic viscosity (μ), each of the respective stress components shall be defined accordingly as

$$\tau_{ij} = \mu\frac{\partial u_i}{\partial x_j} \qquad 1.2(iii)$$

Symmetry holding that $\tau_{ji} = \tau_{ij}$ where $\tau_{ji} = \mu\frac{\partial u_j}{\partial x_i}$ implies that we can also alternatively represent

$$\tau_{ij} = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

And since normal stress τ_{ii} and thermodynamic pressure (p) are both in the x_i-direction, we can express the combined stress tensor (σ_{ij}) in the form

$$\sigma_{ij} = p\delta_{ij} + \tau_{ij} \quad , \qquad \delta_{ij} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \qquad 1.2(iv),$$

so that the total work (dw) done by the secondary flow stresses is then given by

$$dw = \left(dp\,\delta_{ij} + d\,\tau_{ij}\right)dV \qquad 1.2(v)$$

By combining the results in equations (1.2(ii)) and (1.2(iv)), we reduce the energy balance equation for the elementary mass (dm) to

$$dQ = \rho\, d\, V\left\{de + \frac{1}{2} d\left(\vec{V} \bullet \vec{V}\right) + \vec{g} \bullet \vec{H}\right\} + \left(dp\, \delta_{ij} + d\, \tau_{ij}\right) dV$$

But ordinary boundary layer fluid flows take place under conditions where some of the parameters in the last equation are not significant. We assume that the boundary layer flow is both isolated and operating at a constant temperature so that $dQ = 0$ and $d\, e = 0$. We assume further that the boundary layer forms near or on the surface of the main fluid body for which $g dH \approx 0$. With these assumptions we end up with the equation

$$\frac{1}{2} \rho\, d\left(\vec{v} \bullet \vec{v}\right) + \left(dp\, \delta_{ij} + d\, \tau_{ij}\right) = 0 \quad (1.2.1)$$

In component form together with a steady state, we can expand

$$d\left(\vec{v} \bullet \vec{v}\right) = \frac{\partial u_j^2}{\partial x_i} d\, x_i + \frac{\partial u_i^2}{\partial x_j} d\, x_j ,$$

and

$$d\, p = \frac{\partial\, p}{\partial\, x_i} d\, x_i + \frac{\partial\, p}{\partial\, x_j} d\, x_j$$

And from the definition in equation (1.2 (iii)) the viscous drag force above may be expressed as

$$d\, \tau_{ij} = \frac{\partial}{\partial\, x_i}\left(\mu \frac{\partial\, u_i}{\partial\, x_j}\right) \partial\, x_i + \frac{\partial}{\partial\, x_j}\left(\mu \frac{\partial\, u_i}{\partial\, x_i}\right) \partial\, x_j$$

When the last three results are taken into equation (1.2.1), the energy balance equation reduces into its partial derivatives form to give

$$\frac{1}{2} \rho \frac{\partial\, u_j^2}{\partial\, x_i} + \frac{\partial}{\partial\, x_i}\left(\mu \frac{\partial\, u_i}{\partial\, x_j}\right) + \frac{\partial\, p}{\partial\, x_i}\ \delta_{ij} = 0 \qquad (1.2.2a)$$

and

$$\frac{1}{2} \rho \frac{\partial\, u_i^2}{\partial\, x_j} + \frac{\partial}{\partial\, x_j}\left(\mu \frac{\partial\, u_i}{\partial\, x_j}\right) + \frac{\partial\, p}{\partial\, x_j}\ \delta_{ij} = 0 \qquad (1.2.2b)$$

But $\frac{\partial}{\partial\, x_i}\left(\mu \frac{\partial\, u_i}{\partial\, x_j}\right)$ in equation (1.2.2a) is a torsion term involving solid-body rotation. That kind of motion exists only in certain cases of fluid motion such as vortices but not in boundary layer flows. We can correct for this term by either invoking the symmetry property $\tau_{ji} = \tau_{ij}$ right away or alternatively by expressing

$$\frac{\partial\, u_i}{\partial\, x_j} = \frac{1}{2}\left(\frac{\partial\, u_i}{\partial\, x_j} + \frac{\partial\, u_j}{\partial\, x_i}\right) + \frac{1}{2}\left(\frac{\partial\, u_i}{\partial\, x_j} - \frac{\partial\, u_j}{\partial\, x_i}\right).$$

By making the antisymmetric term[3] vanish so that

[3] This term exists only where there is solid body rotation.

$$\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)=0$$

equations (1.2.2a) and (1.2.2b) eventually become identical, which with the coefficient of dynamic viscosity (μ) constant, we get the single governing force equation

$$\frac{1}{2}\rho\frac{\partial u_i^2}{\partial x_j}+\mu\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial p}{\partial x_j}\delta_{ij}=0 \qquad (1.2.3)$$

Equation (1.2.3) is necessarily a two-dimensional one, which is in concurrence with the physical appearance of the ordinary boundary layer flow where we always have only one direction for the main stream and every other direction perpendicular to it reduces to the same 2-dimensional flow problem with the same general solution.

And since the boundary layer flow is not only self-generating but also self-maintaining, it must conserve kinetic energy. So equation (1.2.3) is also subject to the condition that

$$\sum_i u_i^2=u_o^2 \qquad (1.2.4)$$

A final solution to the flow problem is obtained by first evaluating the Kronecker delta in equation (1.2.3) to give the two component equations

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_j}+\upsilon\frac{\partial^2 u_i}{\partial x_j^2}=0 \qquad (1.2.5)^4$$

$$i\neq j$$

and

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_i}+\upsilon\frac{\partial^2 u_i}{\partial x_i^2}+\frac{1}{\rho}\frac{\partial P}{\partial x_i}=0 \qquad (1.2.6)$$

So finally we conclude that the boundary layer flow problem is completely described by equations (1.2.5) and (1.2.6) subject to kinetic energy conservation in equation (1.2.4). And in real space that means an n-dimensional flow shall be represented by 2^n equations.

[4] A synonym of the Hagen-Poisseuille formula

1.3 List of parameter symbols and formulae

The fundamental quantities are mass (m), displacement $(\vec{r})$, distances l and time (t) all primarily in the SI (meter-kilogram-second) Units.

Symbol	Parameter description	Formula or usage
A	Area	$A=l\ x\ l$
e	Internal energy	
E	Mechanical energy	
F, F_o, F_i,	Force	$F = m\frac{d\vec{V}}{dt}$
μ	Coefficient of dynamic viscosity	$\frac{F}{A} = \mu\frac{du}{dy}$
υ	Coefficient of kinematic viscosity	$\upsilon = \frac{\mu}{\rho}$
p	Dynamic pressure	$\frac{F}{A}$
Q,	Heat Energy	
ρ	Fluid density	$\rho = \frac{m}{V}$
Re, Rx	Reynolds number	$R_e = \frac{ul}{\upsilon},\ R_x = \frac{ux}{\upsilon}$
σ, τ, τ_{ij}	Viscous stress	$\tau_{ij} = \mu\frac{\partial u_i}{\partial x_j}$
u, v, u_i, u_j	Velocity components	$u = \frac{dx}{dt},\ v = \frac{dy}{dt},\ u_j = \frac{d x_j}{dt}$
$\vec{V}$	Velocity Vector	$\vec{V} = \frac{d\vec{r}}{dt}$
V	Volume	$V=l\ x\ l\ x\ l$
W	Work done	$W=F\ x\ l$

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PART II

Abstract

This part presents a solution to the boundary layer flow problem developed in part I. The governing equations are simple and the solution is obtained by the process of integration. However, due to the adoption of a two-fluid model and kinetic energy conservation, it has been necessary to abandon the standard Cauchy boundary conditions and employ ordinary limits of integration instead.

Even then, the solution is similarly simple and provides the closest correspondence between theory and practice. The velocity profiles are asymptotic as expected and the corresponding flow profile is the natural energy distribution curve expected of a self-generating and self-maintaining system.

Not only that but previous laws such Bernoulli's law and the Hagen-Poisseuille formula appear spontaneously from the proceedings of the formulation and solution of the fluid dynamical problem.

Key words: boundary layer flow, secondary flow, partial derivatives, integration, kinetic energy conservation, energy distribution curve.

2.1 Formulation of boundary conditions

In Part I it was proposed that the boundary layer flow of a viscous incompressible fluid of density ρ and dynamic viscosity μ is governed by the equation

$$\frac{1}{2}\rho\frac{\partial u_i^2}{\partial x_j}+\mu\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial p}{\partial x_j}\delta_{ij}=0 \qquad (1.2.3)$$

Evaluation of the Kronecker delta in equation (1.2.3) then gave the two component equations

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_j}+\upsilon\frac{\partial^2 u_i}{\partial x_j^2}=0 \qquad (1.2.5)$$

$$i \neq j$$

and

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_i}+\upsilon\frac{\partial^2 u_i}{\partial x_i^2}+\frac{1}{\rho}\frac{\partial P}{\partial x_i}=0 \qquad (1.2.6)$$

Both equations (1.2.5) and (1.2.6) are intergrable over all ranges, but due to kinetic energy conservation we make no presumptions as to the values of parameters on the plate surface. We can only assume that no-slip leads to velocity components u and v that conserve kinetic energy but exact plate surface values are not known. The other extreme boundary is also unattainable because all parameters there behave asymptotically so that we cannot reach any of their limiting values except by closing approximations that are only useful for practical purposes.

So in this kind of situation it is not precisely correct to speak of boundary conditions because there are no boundaries with known values of fluid flow parameters involved. For this reason we shall use limits of integration instead and perform definite integration on the intervals

$$\left.\begin{array}{l}[u_i, u_o], [\partial u_i, 0] \\ \text{and} \\ [u_j, 0], [\partial u_j, 0]\end{array}\right\} \qquad (2.1.1)$$

Even though the above intervals exist their exact coordinates in the flow are not exactly known. What we can say about their locations is that they are within the semi-infinite intervals

$$\left.\begin{array}{l}[x_i, \infty) \\ \text{and} \\ [x_j, \delta_o)\end{array}\right\} \qquad (2.1.2)$$

where δ_o is the boundary layer thickness taken at full plate length. But the boundary layer thickness is also known to grow asymptotically so that its precise size is also not known. Therefore on the intervals (2.1.2) it shall be necessary to use indefinite integration and then thereafter select coordintes desired for the values of the constants of integration.

2.2 Solution to the boundary layer flow problem

Subsequent to the limits of integration given in equation (2.1.1), each of equations (1. 2.5) and (1.2.6) shall have two kinds of solutions, one on the intervals $[u_i, u_o]$ and $[\partial u_i, 0]$ and the other on the intervals $[u_j, 0]$, and $[\partial u_j, 0]$.

Integrating equation (1.2.5) on the intervals $[u_i, u_o]$ and $[\frac{\partial u_i}{\partial x_j}, 0]$ gives

$$\frac{1}{2}\left(u_o^2 - u_i^2\right) = \upsilon \frac{\partial u_i}{\partial x_j}$$

A second integration of the above result involves the variable x_j on the limits [x_j , ∞), which results into an improper integral. This is not physically expedient so we perform the corresponding indefinite integration instead to get

$$x_j + c_1 = \frac{\upsilon}{u_o} \ln\left(\frac{u_o + u_i}{u_o - u_i}\right)$$

Now if we assign some arbitrary value $x_j = x_o$ at which u_i vanishes[5] we get the velocity profile

$$u_j = u_o \frac{e^{\frac{u_o}{\upsilon}(x_j - x_o)} - 1}{e^{\frac{u_o}{\upsilon}(x_j - x_o)} + 1} \qquad (2.2.1a)$$

On the other hand integrating equation (2.1.5) on the intervals [u_i, 0] and [$\frac{\partial u_i}{\partial x_j}$,0] gives

$$-\frac{1}{2} u_i^2 = \upsilon \frac{\partial u_i}{\partial x_j}$$

But we also understand that δ_o is similarly unattainable as a limit so again integration over the interval $[x_j, \delta_o)$ leads to an improper integral, so again we perform the corresponding indefinite integration to get

$$x_j + c_2 = \frac{2\upsilon}{u_i}$$

Now, kinetic energy conservation requires that when the velocity component in (2.2.1a) vanishes; the component transverse to it must becomes u_o, so we have

$$c_2 = \frac{2\upsilon}{u_o} - x_o$$

and consequently the velocity profile

$$u_i = \frac{u_o}{1 + \frac{u_o}{2\upsilon}(x_j - x_o)} \qquad (2.2.2)$$

Observe that once rather than dealing with the limit on the outer δ_o bound of the boundary layer, we got our extreme value from the flat plate side (x_j).

And since every velocity component in equation (2.2.1a) is perpendicular to a corresponding velocity component in equation (2.2.2), we need to make this last equation perpendicular to equation (2.2.1a), and we do so by interchanging the subscripts in equation (2.2.2) to get

[5] Not necessarily on the flat plate surface as is assumed in the no-slip condition

$$u_j = \frac{u_o}{1 + \frac{u_o}{2\upsilon}\left(x_i - x_o\right)} \qquad (2.2.2a)$$

However, without loss of generality, the velocity components in equation (2.2.1a) and (2.2.2) can be plotted on the same axes to give the velocity profiles in figure 2.2.1

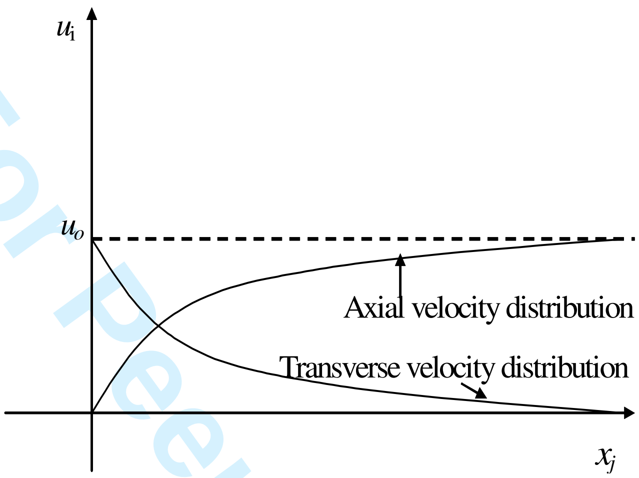

Figure 2.2.1: Boundary layer velocity profiles

The foregoing analysis gives the solution to a typical 2-dimensional boundary layer flow, and if we substitute the velocity components in equations (2.2.1a) and (2.2.2a) into the kinetic energy conservation equation (1.2.4)[6] we get the corresponding two-dimensional flow profile

$$\frac{u_o}{2\upsilon}\left(x_i - x_{oi}\right) + 1 = \frac{e^{\frac{u_o}{2\upsilon}\left(x_j - x_{oj}\right)} + e^{-\frac{u_o}{2\upsilon}\left(x_j - x_{oj}\right)}}{2} \qquad (2.2.3a)$$

The same flow profile is obtained if the expressions in equations (2.2.1a) and (2.2.2a) are subjected to the stream function formulation based on continuity.

We can similarly integrate equation (1.2.6) on the intervals [u_i, u_o] and [$\frac{\partial u_i}{\partial x_j}$, 0] to get

$$\frac{1}{2}\left(u_o^2 - u_i^2\right) - \upsilon\frac{\partial u_i}{\partial x_i} + \frac{1}{\rho}\left(P_{oi} - P_i\right) = 0 \qquad (2.2.4)$$

and on the intervals [u_j, 0] and [$\frac{\partial u_j}{\partial x_j}$, 0] to get

[6] The alternative is to use the stream function formulation

$$-\frac{1}{2}u_j^2-\upsilon\frac{\partial u_j}{\partial x_j}+\frac{1}{\rho}\left(P_{oj}-P_j\right)=0 \qquad (2.2.5)$$

perpendicular to it. We stop here with the notion that equations (2.2.4) and (2.2.5) are statements of Bernoulli's theorem for viscous fluids flowing in the respective directions. If we add the two and invoke kinetic energy conservation we get

$$\upsilon\sum_i\frac{\partial u_i}{\partial x_i}=\frac{1}{\rho}\left(P_o-P\right) \qquad (2.2.6)$$

which is a relationship between pressure and continuity, the existence of one being a necessary condition for the other.

But we can also add equations (2.2.4) and (2.2.5) under continuity to get

$$P_o+\frac{1}{2}\rho u_0^2=P+\frac{1}{2}\rho\sum_j u_j^2 \qquad (2.2.7)$$

which is a statement of the Bernoulli theorem under the condition of constant gravitation imposed earlier in Part I.

The solution above is a two-dimensional one and easily translates into Cartesian (x,y)-coordinates where with $x_i=x$ and $x_j=y$ then $u_i=u$, $u_j=\mathrm{v}$. When translated in terms of these variables equation (2.2.1a) becomes

$$u=u_o\frac{e^{\frac{u_o}{\upsilon}\left(y+y_o\right)}-1}{e^{\frac{u_o}{\upsilon}\left(y+y_o\right)}+1} \qquad (2.2.1b)$$ [7]

whereas equation (2.2.2a) becomes

$$\mathrm{v}=\frac{u_o}{\frac{u_o}{2\upsilon}\left(x-x_o\right)+1} \qquad (2.2.2b)$$

Either continuity under stream function analysis or kinetic energy conservation gives the flow profile

$$\frac{u_o}{2\upsilon}\left(x-x_o\right)+1=\frac{e^{\frac{u_o}{2\upsilon}\left(y+y_o\right)}+e^{-\frac{u_o}{2\upsilon}\left(y+y_o\right)}}{2} \qquad (2.2.3b)$$

Equation (2.2.3b) is the catenary (or the inverse hyperbolic cosine)

$$\frac{x-x_o}{a}=Cosh\left(\frac{y+y_o}{a}\right)-1 \qquad (2.2.3c)$$

$$a=\frac{2\upsilon}{u_o}$$

[7] This solution was first proposed by the author [4] as the best-behaved assumed solution.

in which when all the streamlines corresponding to every point (x_o, y_o) are plotted, we get the flow profile in figure 2.2.2 below.

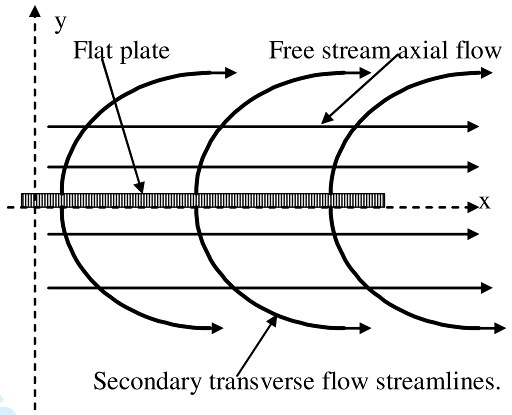

Figure 2.2.2: The boundary layer flow streamline pattern.

Figure 2.2.2 is graphical evidence that the boundary layer flow truly exists as a secondary flow interacting with the main stream. Observe further that (x_o, y_o) is the point where the pumping pressure for the secondary flow is located. And as there is no restriction to the position of this point in relation to the flat plate surface, we have three kinds of flow profiles. There is the completely laminar flow where y_o is negative as in figure 2.2.3(a), we can also have a flow that is transitional between lamina and turbulent whereby $y_o = 0$ as in figure 2.2.3(b), and thirdly we can have a completely turbulent flow in which y_o is positive as in figure 2.2.3(c).

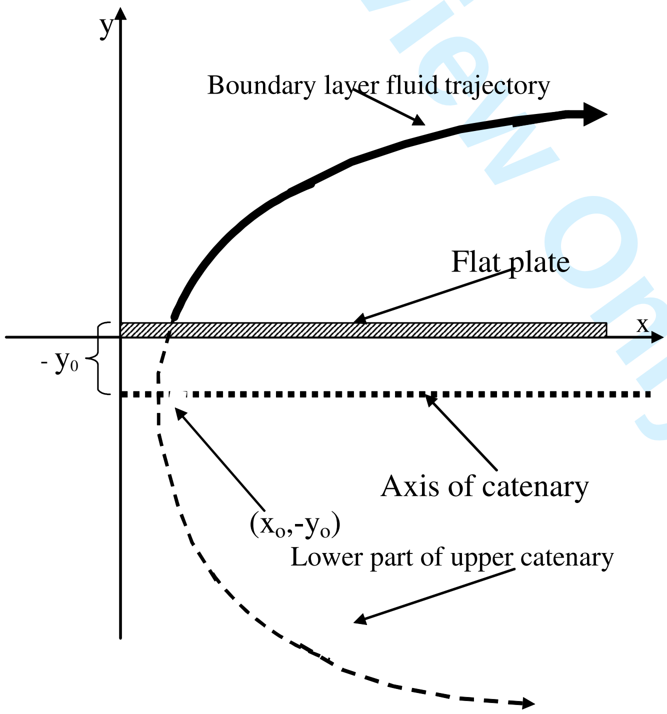

Figure 2.2.3(a): Negative y_o

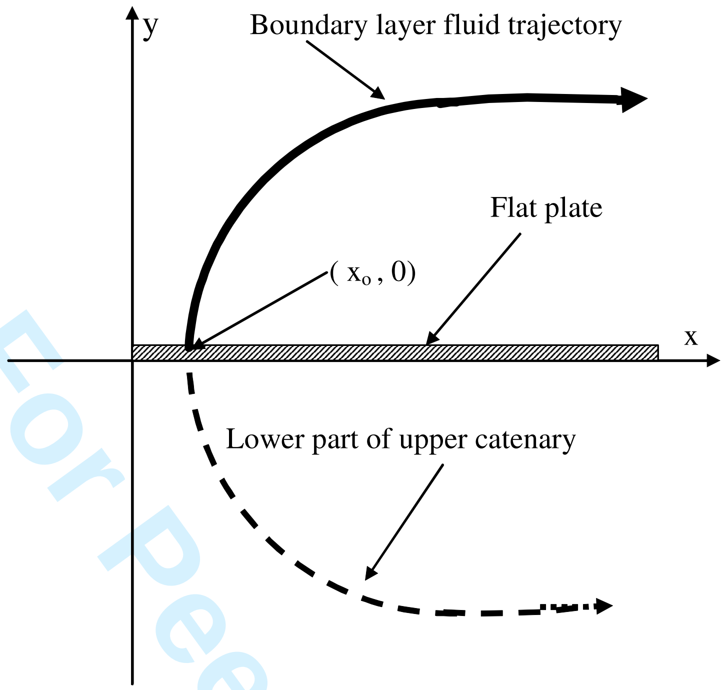

Figure 2.2.3(b): y_o=0

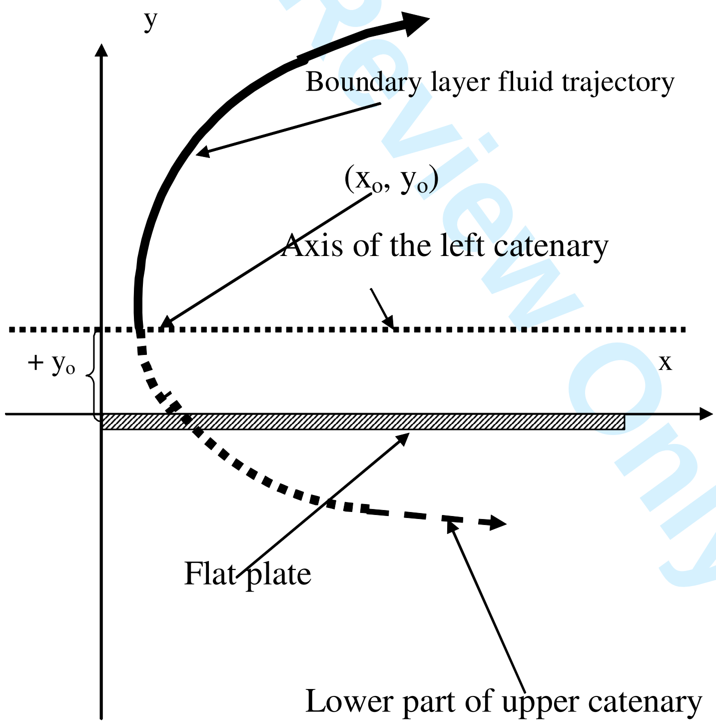

Figure 2.2.3(c): Positive y_o

2.3 List of parameter symbols and formulae

The fundamental quantities are mass (m), displacement $(\vec{r})$, length l and time (t) all primarily in the SI (meter-kilogram-second) Units.

Symbol	Parameter description	Formula or usage
δ_o	Boundary layer thickness at full plate length *l*	$\delta(x) \approx \sqrt{\frac{\upsilon x}{u_O}}$
μ	Coefficient of dynamic viscosity	$\frac{F}{A} = \mu \frac{du}{dy}$
υ	Coefficient of kinematic viscosity	$\upsilon = \frac{\mu}{\rho}$
p	Dynamic pressure	$\frac{F}{A}$
ρ	Fluid density	$\rho = \frac{m}{V}$
u, v, u_i, u_j	Velocity components	$u = \frac{dx}{dt}, \mathrm{v} = \frac{dy}{dt}, u_j = \frac{d\, x_j}{dt}$
V	Volume	$V = l \times l \times l$

References.

1. Currie I. G : Fundamental Mechanics of Fluids, McGraw Hill N:Y (1974)
2. Duncan W.J et al: An Elementary Treatize of Mechanics,Anold London (1968), ELS & Anold (1967).
3. Massey B.S : Mechanics of Fluids 5th ed, Van Nostrand Reinhold U.K (1983).
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A TWO-FLUID MODEL FORMULATION OF THE BOUNDARY LAYER FLOW OF A VISCOUS INCOMPRESSIBLE FLUID PAST A FLAT PLATE AT ZERO INCIDENCE.

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PART III

Abstract

This part presents applications of the solution developed in Part II. A preliminary comment on the change in the physical interpretation of some of the boundary layer flow parameters has been made to point to every possibility of fluid flows having their unique coordinate system.

That is further to the velocity-squared force law and the corresponding drag coefficient for a flat plate being derived right from basic principles and giving results that are comparatively more accurate than the earlier empirical and experimental varieties. And unlike previous versions of the drag coefficient which are known to work for a limited range, an approximation from the current solution operates for an infinite range of Reynolds numbers.

The solution also extends to other profiles such as the flow inside a uniform circular pipe, and the corresponding parameters are in complete agreement with experimental ones.

Key words: Drag force, drag coefficient, uniform circular pipe, real pipe, pseudo-pipe, discharge coefficient, venna contracta.

3.1 Analytical preview

The solution presented in Part II proposes boundary layer flow parameters with a wider physical perspective than the previous ones. Examples include the Reynolds number which is now neither the constant $R_e = \frac{u_o l}{\upsilon}$ representing similar flows nor the variable version $R_x = \frac{u_o x}{\upsilon}$ used for the semi-infinite flat plate. Instead the number stands for every point (X,Y) inside a fluid of kinematic viscosity υ flowing with a uniform velocity u_o such that

$$X = \frac{u_o}{\upsilon} x \quad \text{and} \quad Y = \frac{u_o}{\upsilon} y \qquad (3.1.1)$$

where $(x.y)$ are the rectangular Cartesian coordinates of that point. This newly defined coordinate system is similarly applicable to every other function that corresponding to the fluid flow.

Another boundary layer flow parameter affected in that similar manner is the similarity variable η. This variable is conceived of the scaling argument that boundary layer growth with the downstream coordinate x is given by $\delta(x) \approx \sqrt{\frac{\upsilon x}{u_o}}$.

Using this result one obtains the dimensionless variable $\eta = \frac{y}{\delta(x)}$ expressed variously as $\eta = y\sqrt{\frac{u_o}{\upsilon x}}$ or $\eta = \frac{y}{x}\sqrt{R_x}$. However, this same result can be obtained from the flow profile

$$\frac{u_o}{2\upsilon}(x - x_o) + 1 = \frac{e^{\frac{u_o}{2\upsilon}(y+y_o)} + e^{-\frac{u_o}{2\upsilon}(y+y_o)}}{2} \qquad (2.2.4b)$$

by taking the second order Taylor series approximation of the RHS whereby we get the result

$$\frac{u_o}{2\upsilon}(x - x_o) = \left(\frac{u_O}{2\upsilon}(y + y_o)\right)^2 \qquad (3.1.2)$$

So the similarity variable is actually the quadratic approximation to the boundary layer flow profile.

3.2 Viscous drag on the flat plate

The position of the point (x_O, y_O) relative to the surface of the flat plate surface also affects the size of the viscous drag force. Considering the volume of contact for a two sided flat plate of area A given by $dV = 2Ady$. Equation (1.2.4) then gives the elementary inertial force acting on the plate as[8]

$$dF = -2A\frac{\partial}{\partial y}\left(\frac{1}{2}\rho u^2\right)dy$$

On integration with respect to u on the interval $[u, u_o]$ this latter equation gives the force

$$F = 4A\rho u_0^2 \frac{e^{\frac{u_0}{\upsilon}(y+y_o)}}{\left(e^{\frac{u_0}{\upsilon}(y+y_o)} + 1\right)^2} \qquad (3.2.1)$$

[8] The same result is obtained for $dF = -2A\mu\frac{\partial^2 u}{\partial y^2}dy$ integrated over $\left[\frac{\partial u}{\partial y}, 0\right]$ and $[y, \infty)$

and the total drag force exerted on the plate surface obtained by taking y = 0 is

$$F_o = 4A\rho u_0^2 \frac{e^{\frac{u_0}{\upsilon} y_o}}{\left(e^{\frac{u_0}{\upsilon} y_o} + 1\right)^2} \qquad (3.2.2)$$

Equation (3.2.2) is a proportionality to the square of the free stream velocity as expected and from the definition [2] $C_D = \frac{F}{\frac{1}{2} A\rho u_o^2}$ the corresponding drag coefficient

$$C_D = 8 \frac{e^{\frac{u_0}{\upsilon} y_0}}{\left(e^{\frac{u_0}{\upsilon} y_0} + 1\right)^2} \qquad (3.2.3)$$

is a normal distribution represented by the curve in figure 3.2.1 below.

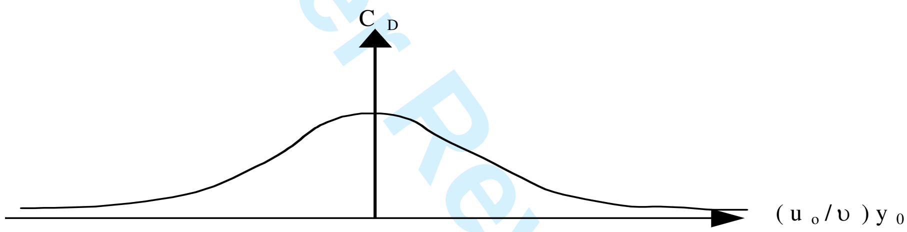

Figure 3.2.1: C_D from equation (3.2.3)

Few previuos empirical or experimental formulae have ever been close to equation (3.2.3). Stokes drag coefficient[3]

$$C_D = \frac{24}{R_e}$$

is known to work only for Reynolds numbers Re<0 whereas Carl Wilhem Oseen's [3] first approximation

$$C_D = \frac{24}{R_e}\left(1 + \frac{3}{16} R_e\right)$$

is valid only for Re<1. Oseen's improved approximation

$$C_D = \frac{24}{R_e}\left(1 + \frac{3}{16} R_e\right)^{\frac{1}{2}}$$

works no farther than Re=100 and no suitable approximate formulae has been suggested for the Reynolds numbers Re>1000. Over that range the drag coefficient for a flat plate no longer depends on the Reynolds number and values of C_D had to be plotted against the ratio of plate-length/plate-breadth instead [3] to give the distribution in table 3.2.1.

Length/Breadth	1	2	4	10	18	∞
C_D	1.10	1.15	1.19	1.29	1.40	2.01

Table 3.2.1: Experimental values of C_D for Re>1000

On the other hand the accuracy of equation (3.2.3) extends further to its second order Taylor approximation

$$C_D = \frac{2}{\left(1 + \frac{1}{8}\left(\frac{u_o}{\upsilon} y_o\right)^2\right)^2} \qquad (3.2.4(a))$$

which with the Reynolds number redefined to include $R_e = \frac{u_0}{\upsilon} y_0$ gives the equivalent alternative form

$$C_D = \frac{2}{\left(1 + \frac{1}{8} R_e^2\right)^2} \qquad (3.2.4(b))$$

Equation (3.2.4(b)) gives the maximum value $C_D = 2$ at Re=0 as predicted by the tabular experimental results above.

Furthermore, equation (3.2.4(b)) is valid over the whole range $-\propto < R_e < +\propto$ and a plot of it in figure 3.2.2 compares to Duncan's [2] similar plot in figure 3.2.3. The difference between the two plots is only dimensional where Duncan uses $Log_{10} R_e = \frac{u_o c}{\upsilon}$ as the independent variable (c is some variable Duncan refers to as the chord length of the flat plate) instead of the same cordinate used in equation (3.2.4(a)) but defined in terms of equation (3.1.1).

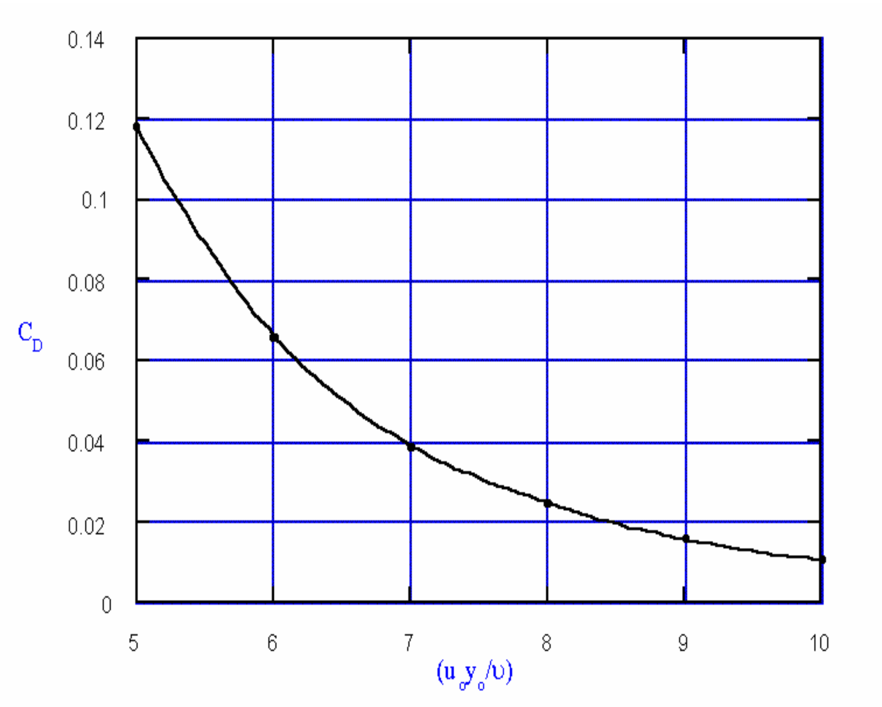

Figure 3.2.2: C_D from equation (3.2.4(a))

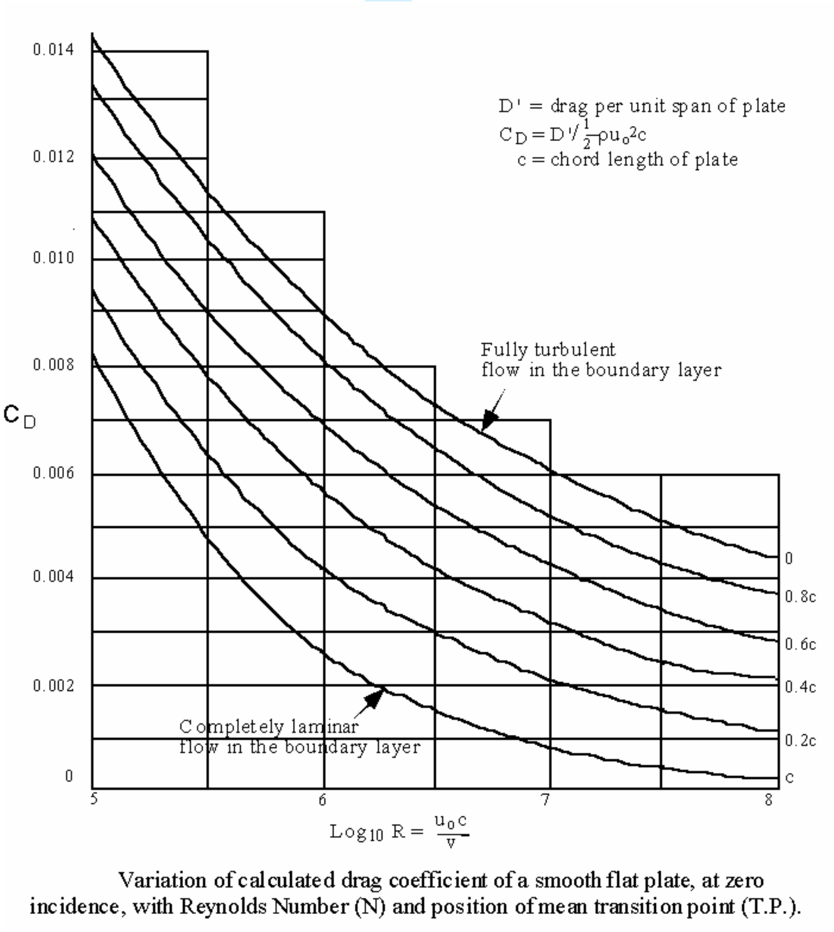

Figure 3.2.3: Duncan's Experimental C_D Curves [2]

3.3 Application to the uniform circular pipe

The point (x_o,y_o) similarly exists in the flow through a uniform circular pipe. The pipe then appears to consist of two cylinders, the inner real pipe of radius r_o surrounded by an outer pseudo-pipe of radius $R_o = r_o + y_o$ as shown in figure 3.3.1.

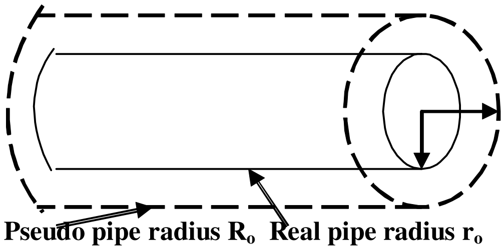

Figure 3.3.1: Pseudo pipe surrounding the real pipe

Assume that at steady flow the viscous drag force acting inside the pipe is constant so that

$$\mu \frac{d^2 u}{d r^2} = -C_0 \qquad 3.3.1$$

A first indefinite integration of equation (3.3.1) with $\frac{du}{dr} = 0$ at r = 0 gives

$$\mu \frac{du}{dr} = -C_0 r$$

A second integration with u = 0 at r = R_o and u = u_o at r = 0 gives the velocity distribution

$$u = \frac{u_o}{R_o^2}\left(R_o^2 - r^2\right) \qquad 3.3.2$$

that corresponds to the quadratic velocity profile in figure 3.3.2 below.

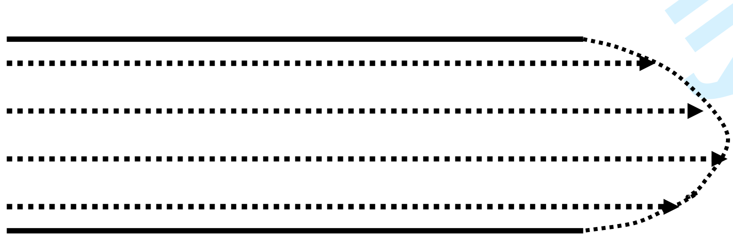

Figure 3.3.2: The Hagen-Poiseuille velocity profile

For the velocity distribution in equation (3.3.2), the discharge of the pipe is given by the integral of

$$dQ = 2\pi\, r\, u\, d\, r$$

over the range $r=0$ *to* $r = R_o$ to give

$$Q = \frac{u_o \pi r_0^2}{2R_0^2}\left(2R_0^2 - r_0^2\right) \qquad 3.3.3$$

With the ideal discharge

$$Q_0 = \pi r_o^2 u_o$$

and small values of y_o compared to r_o, we get the discharge coefficient

$$C_d = \frac{1}{2} + \frac{r_o y_o}{\left(r_o + y_o\right)^2} \qquad 3.3.4$$

That the discharge always lies between 0.6% and 0.65% [3] shows that it is practically not possible to get rid of y_o. By assigning the proportionality $r_o = ky_o$ we deduce from equation (3.3.4) that the maximum discharge cannot exceed 0.75%, a result that has been confirmed by experiment.

From steady state and the input velocity being equal to the output velocity, we should have expected the average flow velocity

$$u_p = \frac{u_o}{2R_o^2}\left(2R_o^2 - r_o^2\right) \qquad 3.3.5$$

But high pressure secondary flow at the points (x_o, y_o) overriding the main flow at the pipe outlet forms the venna contracta as shown in figure 3.3.3.

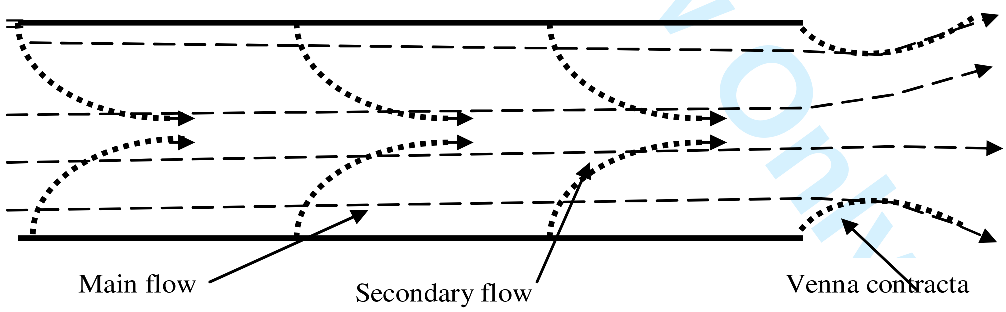

Figure 3.3.3: The secondary flow model velocity profile

3.4 List of parameter symbols and formulae

The fundamental quantities are mass (m), displacement $(\vec{r})$, distances l and time (t) all primarily in the SI (meter-kilogram-second) Units.

Symbol	Parameter description	Formula or usage
A	Area	$A = l \times l$
C_d	Fluid discharge coefficient	$C_d = \frac{Q}{Q_o}$
C_D	Fluid drag coefficient	$C_D = \frac{F}{\frac{1}{2} A\rho u_o^2}$
F, F_o, F_i,	Force	$F = m\frac{d\vec{V}}{dt}$
μ	Coefficient of dynamic viscosity	$\frac{F}{A} = \mu\frac{du}{dy}$
υ	Coefficient of kinematic viscosity	$\upsilon = \frac{\mu}{\rho}$
p	Dynamic pressure	$\frac{F}{A}$
Q, Q_o	Fluid discharge from a circular pipe	$Q_0 = \pi r_o^2 u_o$
ρ	Fluid density	$\rho = \frac{m}{V}$
R_o, r, r_o	Pipe radius	$R_o = r_o + y_o$
Re, Rx	Reynolds number	$R_e = \frac{ul}{\upsilon}$, $R_x = \frac{ux}{\upsilon}$
u, v, u_i, u_j	Velocity components	$u = \frac{dx}{dt}, \mathrm{v} = \frac{dy}{dt}, u_j = \frac{dx_j}{dt}$
V	Volume	$V = l \times l \times l$

References.

1. Currie I. G : Fundamental Mechanics of Fluids, McGraw Hill N:Y (1974)

2. Duncan W.J et al: An Elementary Treatize of Mechanics,Anold London (1968), ELS & Anold (1967).

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6. Coburn N: Vector and Tensor Analysis; McGraw Hill Inc. (1955).