A TWO-FLUID MODEL FORMULATION OF THE BOUNDARY LAYER FLOW OF A VISCOUS INCOMPRESSIBLE FLUID PAST A FLAT PLATE AT ZERO INCIDENCE.

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Abstract

This boundary layer flow formulation is based on a two fluid model, the main flow parallel to the flat plate surface and a secondary flow transverse to it. And due to an apparent fault of the momentum operator, the energy conservation equation $\Delta Q = \Delta E + \Delta W$ has been employed to alternatively formulate the corresponding flow problem. Subsequent thereto a set of two kinds of governing equations is obtained, the first relating to shear stresses in a manner similar to the Hagen-Poiseuille treatment of the flow inside a circular pipe, and the second relating to normal stresses in the like manner of Bernoulli's law. Additional governing conditions include kinetic energy conservation only.

Even though the solution to the above formulation appears much too simple, it provides results that closely correspond to previously determined empirical and experimental ones. Velocity profiles are asymptotic in real space and the corresponding flow profile is the natural energy distribution curve expected of a self-generating and self-maintaining system. Moreover, experimental laws such as the velocity-squared force law and the corresponding drag coefficient are obtained from the solution with exact results. Not only that but the solution also extends to other related fluid flow problems such as the flow inside a uniform circular pipe for which the corresponding parameters are in complete agreement with experimental results.

Key words: Boundary layer flow, primary flow, secondary flow, viscous incompressible fluids, kinetic energy conservation, energy balance.

1.1 A brief comparative analysis

Secondary fluid flow formation is a common phenomenon associated with the drag effects of boundary layers on solid bodies immersed in fluid flows [2]. It is also known to occur when a fluid flows around a bend in which case it is assumed to be caused by the centrifugal force arising from circulation. Whatever the cause, a secondary flow is a stream of fluid that flows from a flat solid surface that is immersed parallel to a main fluid stream. Such a property gives the secondary flow its own identity complete with its own component of velocity transverse to the flat surface. The boundary layer flow is then understood to consist of the region of the subsequent interaction between the main stream and the secondary stream as shown in figure 1.1.1. But due to the presence of viscous drag effects, the interaction between the two flows is effectively that of the main stream dragging the secondary flow streamlines along. This is essentially the basis of the two-fluid model description of the boundary layer flow where with secondary flow velocities u parallel to the solid surface and V transverse to it, the boundary layer flow then acquires the profile shown in figure 1.1.2.

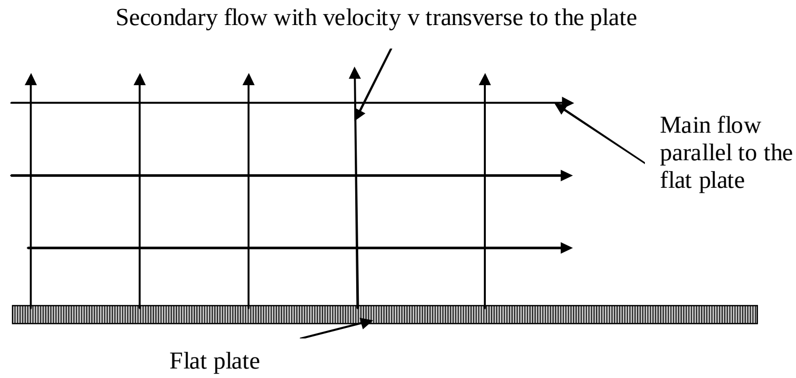

Figure 1.1.1: Ideal secondary flow formation

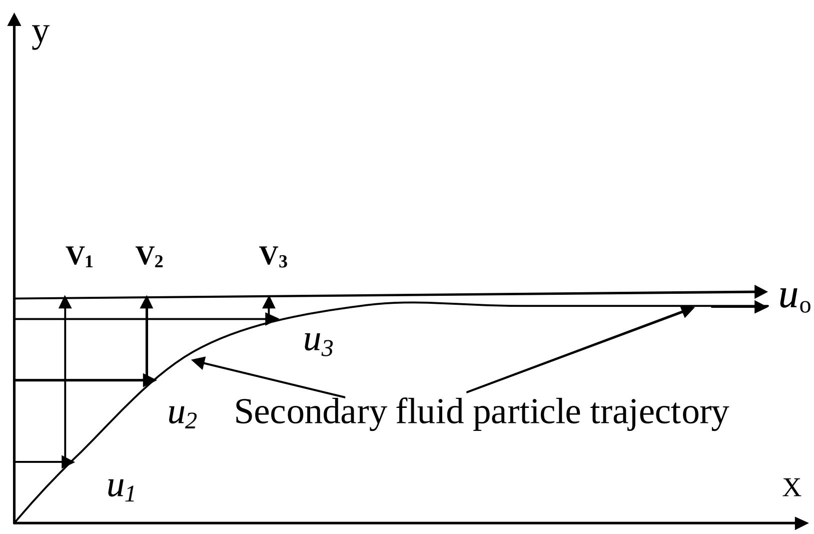

Figure 1.1.2: Secondary flow profile under viscous action

The consequences of the two-fluid model profile in figure 1.1.2 above include the following basic observations:

- No-slip taken together with kinetic energy conservation $u^2 + V^2 \equiv u_o$ require that when $u = 0$ then $V = u_o$, so that V does not vanish on the plate surface.
- The maximum transverse velocity V_{max} is on the plate surface and decreases upwards as the secondary stream becomes absorbed into the main stream.
- Due to viscous drag effects the velocity profiles for both the main flow and the secondary flow exhibit changes in velocity perpendicular to their respective flow

directions so that quantities like $\frac{\partial u}{\partial y}$.and $\frac{\partial \mathrm{v}}{\partial \mathrm{x}}$ carry a particularly significant role in the subsequent formulae.

These observations have two significantly negative repercussions; that is, whereas the first and second observations are at variance with the Cauchy boundary conditions, the third observation does not fit into the definition of momentum, the latter which must be in the direction of the corresponding change in velocity. So taken together the three observations constitute an obvious suggestion that the two-fluid model boundary layer flow problem may not easily respond to the Navier-Stokes (N-S) momentum formulation of the fluid flow problem. This possibility is further accentuated by the fact that the momentum operator ($\vec{\mathrm{V}}\bullet\vec{\nabla}$) used to derive the N-S equations is inherently limited to the derivation and solution of rectilinear solid body motion.

Consider a resultant force $(\vec{F_i})$ acting on a mass (m) that is moving with an instantaneous velocity $(\vec{V}(\mathrm{t}))$ so that Newton's second law of motion gives

$$\vec{F_i} = m\frac{d\vec{V}}{dt}$$

From it we obtain two different results, both of which exist in theory but one of which may not exist in practice. If we subject the RHS of the above force to partial differentiation in space and time we get the force

$$\vec{F}_1 = m\left\{\frac{\partial\vec{V}}{\partial\,\mathrm{t}} + (\vec{V}\bullet\vec{\nabla})\vec{V}\right\} \qquad (1.1.1)$$

Alternatively we can postpone the partial differentiation and first calculate the work done

$$\vec{F_i}\bullet d\vec{r} = \frac{1}{2}m\,d\left(\vec{V}\bullet\vec{V}\right)$$

And if thereafter we apply a partial differentiation similar to the previous one we have

$$\vec{F_2}\bullet\vec{V} = \vec{V}\bullet m\frac{\partial\,\vec{V}}{\partial\,\mathrm{t}} + \vec{V}\bullet\frac{1}{2}m\vec{\nabla}(\vec{V}\bullet\vec{V})$$

from which we get the expression for the corresponding force

$$\vec{F_2} = m\frac{\partial\,\vec{V}}{\partial\,\mathrm{t}} + \frac{1}{2}m\vec{\nabla}(\vec{V}\bullet\vec{V}) \qquad (1.1.2)$$

But vector algebra requires that for any two vectors $\vec{A}$ and $\vec{B}$

$$\vec{\nabla}\left(\vec{A}\bullet\vec{B}\right) = \left(\vec{B}\bullet\vec{\nabla}\right)\vec{A} + \left(\vec{A}\bullet\vec{\nabla}\right)\vec{B} + \vec{B}\times\left(\vec{\nabla}\times\vec{A}\right) + \vec{A}\times\left(\vec{\nabla}\times\vec{B}\right)$$

where with $\vec{A}=\vec{V}$ and $\vec{B}=\vec{V}$ we have

$$\vec{\nabla}\left(\vec{V}\bullet\vec{V}\right) = 2\left(\vec{V}\bullet\vec{\nabla}\right)\vec{V} + 2\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right) \qquad (1.1.3)$$

From equation (1.1.3) it follows that

$$\frac{1}{2}\nabla\left(\vec{V}\bullet\vec{V}\right)\neq\left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}$$

except when either $\vec{V}=0$ or $\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)=0$. That is, $\vec{F}_1$ and $\vec{F}_2$ shall be equal if and only if either both do not exist or both are irrotational, otherwise $\vec{F}_1\neq\vec{F}_2$. But the secondary flow streamline in figure 1.1.2 shows that a boundary layer flow is always rotational and $\vec{V}\times\left(\vec{\nabla}\times\vec{V}\right)\neq 0$ [1]. And since $\vec{F}_1$ lacks the rotational term, it is therefore incapable of formulating the trajectory of the secondary fluid shown in figure 1.1.2.

The difference between $\vec{F}_1$ and $\vec{F}_2$ is further confirmed by their corresponding work-energy relations, that is,

$$F_1\bullet\vec{V}=\frac{1}{2}m\frac{\partial\ \vec{V}\bullet\vec{V}}{\partial\ \mathrm{t}}+\ m\vec{V}\bullet\vec{\nabla}(\ \vec{V}\bullet\vec{V})$$

whereas

$$\overrightarrow{F_2}\bullet\vec{V}=\frac{1}{2}m\frac{\partial\left(\vec{V}\bullet\vec{V}\right)}{\partial\ \mathrm{t}}+\frac{1}{2}m\vec{V}\bullet\vec{\nabla}(\vec{V}\bullet\vec{V})$$

Further still, even though both of the quantities $\nabla(\overrightarrow{V}\bullet\overrightarrow{V})$ and $(\vec{V}\bullet\vec{\nabla})\ \vec{V}$ have the dimensions of force, the former corresponds to a regular definition of a Newtonian force whereas the latter has no Newtonian physical equivalent. So $\vec{F}_1$ may actually not exist as a physical force and the only choice left open for us is to adopt the force $\vec{F}_2$.

1.2 The Governing equations

Simple analysis shows that the force $\vec{F}_2$ (from equation (1.1.2)) cannot directly replace $\vec{F}_1$ (from equation (1.1.1)) in the formula for the Navier-Stokes equations. So as to abandon $\vec{F}_1$ in favor of $\vec{F}_2$, we must completely restructure the momentum formula and the corresponding Navier-Stokes equations. But it has been shown above that $\vec{F}_1$ is actually contained in $\vec{F}_2$ [2], so any governing formulation obtained using $\vec{F}_2$ must similarly contain some, if not all the ingredients of $\vec{F}_1$. Therefore, the best way to resolve the apparent conflict of the application of the two kinds of forces is to find a common ground in which both exist.

Accordingly, we recall that except for the factor $\frac{1}{2}$ both $\vec{F}_1$ and $\vec{F}_2$ have the same work-energy relations, that is,

[1] The secondary flow streamline approaching the free stream asymptotically testifies to this fact.

[2] Previously $\vec{F}_1$ has successfully been employed to solve a similar boundary layer problem.

$$F_1 \bullet \vec{V} = \frac{1}{2} m \frac{\partial \ \vec{V} \bullet \vec{V}}{\partial \ \mathrm{t}} + \ m\vec{V} \bullet \vec{\nabla}(\ \vec{\nabla} \bullet \vec{V})$$

and

$$\overrightarrow{F_2} \bullet \vec{V} = \frac{1}{2} m \frac{\partial \ \left(\vec{V} \bullet \vec{V}\right)}{\partial \ \mathrm{t}} + \frac{1}{2} m\vec{V} \bullet \vec{\nabla}(\vec{V} \bullet \vec{V})$$

So basing our argument on this finding we can strongly assert that $\vec{F}_1$ can be successfully replaced by $\vec{F}_2$ if we re-formulate our fluid flow problem in terms of energy. And one way of achieving this is to invoke the first law of thermodynamics and subsequently use energy balance [2][3] as our boundary layer flow governing condition.

The basic form of the energy conservation law states that the change in the total energy (E) is equal to the sum of the change in the heat energy (Q) and the work (W) done on[3] the system. Using the symbol of change Δ, one alternative way of expressing this law is

$$\Delta E = \Delta Q - \Delta W \qquad 1.2\,(\mathrm{i})$$

(The negative sign accompanying the work signifies the output work done by the system as compared to an input work done on the system).

An elementary mass (dm) of secondary fluid of density (ρ) having an elementary volume (dV) is given by $dm = \rho \ dV$. And if a unit mass of the same fluid flowing with velocity $\left(\vec{V}\right)$ under a total head $\left(\vec{H}\right)$ possesses internal energy (de), translational kinetic energy $\left(\frac{1}{2} d\left(\vec{V} \bullet \vec{V}\right)\right)$ and gravitational potential $\left(\vec{g} \bullet \vec{H}\right)$, the elementary mass (dm) then possesses the total energy (dE) given by

$$dE = \rho \ dV\left(de + \frac{1}{2} d\left(\vec{V} \bullet \vec{V}\right) + \vec{g} \bullet \vec{H}\right) \qquad 1.2(\mathrm{ii})$$

The viscous drag between the main flow and the secondary flow streamline in figure 1.1.2 is the same as that between flat fluid planes in relative motion. Under such consideration, two planes moving with relative velocity (∂u_i) in the x_i-direction experience two stresses, τ_{ii} in the x_i-direction and τ_{ij} in the x_j-direction. Taken with the coefficient of dynamic viscosity (μ), each of the respective viscous stress components above shall be defined accordingly as

$$\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} \qquad 1.2(\mathrm{iii})$$

[3] Sometimes it is the work done by the system that is considered.

Symmetry holding that $\tau_{ji} = \tau_{ij}$ where $\tau_{ji} = \mu \frac{\partial u_j}{\partial x_i}$ implies that we can also alternatively represent

$$\tau_{ij} = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

And since normal stress τ_{ii} and thermodynamic pressure (p) are both in the x_i-direction, we can express the combined stress tensor (σ_{ij}) in the form

$$\sigma_{ij} = p\delta_{ij} + \tau_{ij} \quad , \qquad \delta_{ij} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases} \qquad 1.2(\text{iv}),$$

so that the total work (dw) done by the secondary flow stresses is then given by

$$dw = (dp\,\delta_{ij} + d\,\tau_{ij})dV \qquad 1.2(\text{v})$$

Combining the results in equations (1.2(ii)) and (1.2(v)) reduces the energy balance equation for the elementary mass (dm) to

$$dQ = \rho\, d\, V\left\{de + \frac{1}{2}d\left(\vec{V}\bullet\vec{V}\right) + \vec{g}\bullet\vec{H}\right\} + \left(dp\,\delta_{ij} + d\,\tau_{ij}\right)dV$$

But ordinary boundary layer fluid flows take place under conditions where some of the parameters in the last equation are not significant. We assume that the boundary layer flow is both isolated and operating at a constant temperature so that $dQ = 0$ and $d\,e = 0$. We assume further that the boundary layer forms near or on the surface of the main fluid body for which $gdH \approx 0$. With these assumptions we end up with the equation

$$\frac{1}{2}\rho\, d\left(\vec{V}\bullet\vec{V}\right) + \left(dp\,\delta_{ij} + d\,\tau_{ij}\right) = 0 \qquad (1.2.1)$$

In component form we make the following expansions relative to the x_j direction

$$\left.\begin{aligned} d\left(\vec{V}\bullet\vec{V}\right) &= \frac{\partial u_i^2}{\partial x_j} d x_j \\ d\,p &= \frac{\partial p}{\partial x_j} d x_j \\ d\,\tau_{ij} &= \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) d\,x_j \end{aligned}\right\} \qquad (1.2.2)$$

By substituting the equivalent parameters from equation (1.2.2) into equation (1.2.1) at constant coefficient of dynamic viscosity (μ), we get the single[4] governing force equation

$$\frac{1}{2}\rho\frac{\partial u_i^2}{\partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial p}{\partial x_j}\delta_{ij} = 0 \qquad (1.2.3)$$

[4] This is in fact two force equations in two perpendicular directions.

Equation (1.2.3) is necessarily a two-dimensional one in concurrence with the physical properties of the ordinary boundary layer flow where we always have only one direction for the main stream, and every other direction perpendicular to it reduces to the same 2-dimensional flow problem with the same general solution.

Finally, the boundary layer flow problem may be further simplified by evaluating the Kronecker delta in equation (1.2.3) to give the two component equations

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_j}+\upsilon\frac{\partial^2 u_i}{\partial x_j^2}=0 \qquad (1.2.4)$$

$$i \neq j$$

and

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_i}+\upsilon\frac{\partial^2 u_i}{\partial x_i^2}+\frac{1}{\rho}\frac{\partial P}{\partial x_i}=0 \qquad (1.2.5)$$

On the other hand, the boundary layer flow is not only self-generating but it is also self-maintaining. That is, as soon as a flat plate is immersed in the main fluid stream, a secondary flow immediately comes up and continues to exist as long as the plate remains in place. Such behaviour is inherent of a system that conserves kinetic energy, so

$$\sum_i u_i^2 = u_o^2 \qquad (1.2.6)$$

must be an added condition so that in the end we may conclude that the boundary layer flow problem is completely described by equations (1.2.4) and (1.2.5) and (1.2.6). That way an n-dimensional flow shall be represented by 2^n +1 governing equations, that is, 2^n force equations plus one kinetic energy conservation equation.

2.1 Formulation of boundary conditions

It is proposed above that the boundary layer flow of a viscous incompressible fluid of density ρ and dynamic viscosity μ is governed by the equation

$$\frac{1}{2}\rho\frac{\partial u_i^2}{\partial x_j}+\mu\frac{\partial^2 u_i}{\partial x_j^2}+\frac{\partial p}{\partial x_j}\delta_{ij}=0 \qquad (1.2.3)$$

On evaluation of the Kronecker delta this gave rise to the two component equations

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_j}+\upsilon\frac{\partial^2 u_i}{\partial x_j^2}=0 \qquad (1.2.4)$$

$$i \neq j$$

and

$$\frac{1}{2}\frac{\partial u_i^2}{\partial x_i}+\upsilon\frac{\partial^2 u_i}{\partial x_i^2}+\frac{1}{\rho}\frac{\partial P}{\partial x_i}=0 \qquad (1.2.5)$$

Both equations (1.2.4) and (1.2.5) are intergrable over all ranges, but due to kinetic energy conservation we make no presumptions as to the values of the parameters on the plate surface. We can only assume that no-slip leads to kinetic-energy-conserving velocity components u_S parallel to the plate surface and v_S transverse to it, but exact values are not known. The other extremity of the boundary layer is also unknown because all parameters there behave asymptotically so that we cannot reach any of their limiting values, we get varying approximate values for practical use depending on the degree of accuracy required by the coresponding calculation.

In this kind of situation it is not precisely correct to speak of boundary conditions because there are no boundaries with known values of fluid flow parameters involved. For this reason we shall use limits of integration instead and perform definite integration on the intervals

$$\left.\begin{array}{l} [u_i, u_o], [\partial u_i, 0] \\ \text{and} \\ [u_i, 0], [\partial u_i, 0] \end{array}\right\} \qquad (2.1.1)$$

These limits relate to the velocity component parallel to the flat plate which in the end becomes the free stream velocity u_o and the velocity component transverse to the flat plate which eventually vanishes. Noting that there is no transverse flow within the free stream, the derivatives of both velocities must vanish at the free stream side of the flow.

But even though the above intervals exist, their exact space coordinates are also not known. What we can say about their locations is that they are within the respective intervals

$$\left.\begin{array}{l} [x_j, \infty) \\ \text{and} \\ [x_j, \delta_o) \end{array}\right\} \qquad (2.1.2)$$

where δ_o is the boundary layer thickness taken at full plate length. But the boundary layer thickness is also known to grow asymptotically so that its exact size is also not measurable. Therefore on the intervals (2.1.2) it shall be necessary to use indefinite integration and thereafter select coordintes for which values of desired constants of integration exist.

2.2 Solution to the boundary layer flow problem

Using the limits of integration given in equation (2.1.1) each of equations (1. 2.4) and (1.2.5) shall have two kinds of solutions, one on the intervals $[u_i, u_o]$ and $[\partial u_i, 0]$ and the other on the intervals $[u_i, 0]$, and $[\partial u_i, 0]$.

Integrating equation (1.2.4) on the intervals $[u_i,\ u_o]$ and $[\frac{\partial u_i}{\partial x_j},\ 0]$ gives

$$\frac{1}{2}\left(u_o^2-u_i^2\right)=\upsilon\frac{\partial u_i}{\partial x_j}$$

But the integration of the latter result involves the variable x_j which is on the interval [x_j, ∞), which requires the use of an improper integral. This is not physically expedient so we perform the corresponding indefinite integration instead to get

$$x_j+c_1=\frac{\upsilon}{u_o}\ln\left(\frac{u_o+u_i}{u_o-u_i}\right)$$

Then, if we assign some arbitrary value $x_j = x_o$ as the physical point at which u_i vanishes[5] we get the axial velocity profile

$$u_i=u_o\frac{e^{\frac{u_o}{\upsilon}\left(x_j-x_o\right)}-1}{e^{\frac{u_o}{\upsilon}\left(x_j-x_o\right)}+1} \qquad (2.2.1a)$$

On the other hand integrating equation (2.1.4) on the intervals [u_i, 0] and [$\frac{\partial u_i}{\partial x_j}$,0] gives

$$-\frac{1}{2}u_i^2=\upsilon\frac{\partial u_i}{\partial x_j}$$

Since δ_o is similarly not mathematically attainable as a limit, integration over the interval $[x_j,\ \delta_o)$ leads us to a purely hypothetical result, so rather than have δ_o as our upper limit of integration, again we perform the corresponding indefinite integration to get

$$x_j+c_2=\frac{2\upsilon}{u_i}$$

Now, kinetic energy conservation requires that when the velocity component in (2.2.1a) vanishes; the component transverse to it must becomes u_o, so we have

$$c_2=\frac{2\upsilon}{u_o}-x_o$$

and consequently we get the transverse velocity profile

[5] Not necessarily on the flat plate surface as is assumed in the no-slip condition

$$u_i = \frac{u_o}{1 + \frac{u_o}{2\upsilon}\left(x_j - x_o\right)}$$

Note here that rather than get values on the δ_o side of the boundary layer as should have been expected, we got our limiting values on the (x_o) side instead.

And since every velocity component in equation (2.2.1a) is perpendicular to the corresponding transverse component given in the latter velocity distribution, we need to make this last equation perpendicular to equation (2.2.1a), and we do so by interchanging the subscripts in the latter to get

$$u_j = \frac{u_o}{1 + \frac{u_o}{2\upsilon}\left(x_i - x_o\right)} \qquad (2.2.2a)$$

However, without loss of generality, the two velocity components can be plotted on the same axes to give the velocity profiles in figure 2.2.1

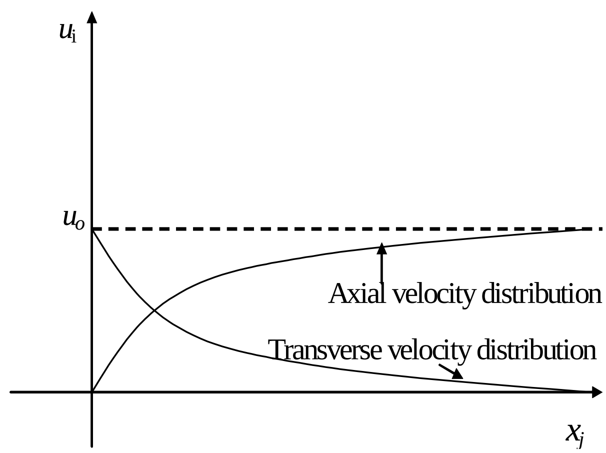

Figure 2.2.1: Boundary layer velocity profiles

The foregoing analysis provides the velocity profiles for a typical 2-dimensional boundary layer flow, and if we substitute the velocity components in equations (2.2.1a) and (2.2.2a) into the kinetic energy conservation equation (1.2.6)[6] we get the corresponding two-dimensional flow profile

[6] The alternative is to use the stream function formulation

$$\frac{u_o}{2\upsilon}\left(x_i - x_{oi}\right) + 1 = \frac{e^{\frac{u_o}{2\upsilon}\left(x_j - x_{oj}\right)} + e^{-\frac{u_o}{2\upsilon}\left(x_j - x_{oj}\right)}}{2} \quad (2.2.3a)$$

The same flow profile is obtained if the expressions in equations (2.2.1a) and (2.2.2a) are subjected to the stream function formulation based on continuity.

The solution above easily translates into Cartesian (x,y)-coordinates where with $x_i = x$ and $x_j = y$ and $u_i = u$, $u_j = \mathrm{v}$, equation (2.2.1a) gives

$$u = u_o \frac{e^{\frac{u_o}{\upsilon}\left(y - y_o\right)} - 1}{e^{\frac{u_o}{\upsilon}\left(y - y_o\right)} + 1} \quad (2.2.1b)$$ [7]

whereas equation (2.2.2a) becomes

$$\mathrm{v} = \frac{u_o}{\frac{u_o}{2\upsilon}\left(x - x_o\right) + 1} \quad (2.2.2b)$$

These latter two profiles under either stream function analysis or kinetic energy conservation give the flow profile

$$\frac{u_o}{2\upsilon}\left(x - x_o\right) + 1 = \frac{e^{\frac{u_o}{2\upsilon}\left(y - y_o\right)} + e^{-\frac{u_o}{2\upsilon}\left(y - y_o\right)}}{2} \quad (2.2.3b)$$

Equation (2.2.3b) is the standard natural energy distribution curve called the inverse hyperbolic cosine or the catenary

$$\frac{x - x_o}{a} = Cosh\left(\frac{y - y_o}{a}\right) - 1 \quad (2.2.3c)$$

$$a = \frac{2\upsilon}{u_o}$$

in which when all the streamlines corresponding to every point (x_o, y_o) are plotted, we get the flow profile in figure 2.2.2 below.

[7] This solution was first proposed by the author [4] as the best-behaved assumed solution.

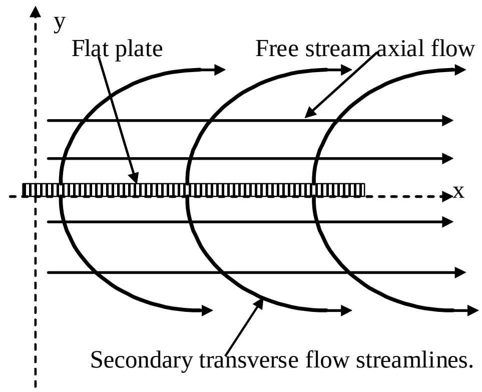

Figure 2.2.2: The boundary layer flow streamline pattern.

Figure 2.2.2 is actually the graphical representation of the boundary layer flow with a very clear manifestation of the secondary flow interacting with the main stream. It is hereby further postulated that (x_o,y_o) is the point where the pumping pressure for the secondary flow is located.

And as there is no restriction to the position of the point (x_o,y_o) in relation to the flat plate surface, we get three kinds of flow profiles. There is the completely laminar flow where y_o is negative as in figure 2.2.3(a).

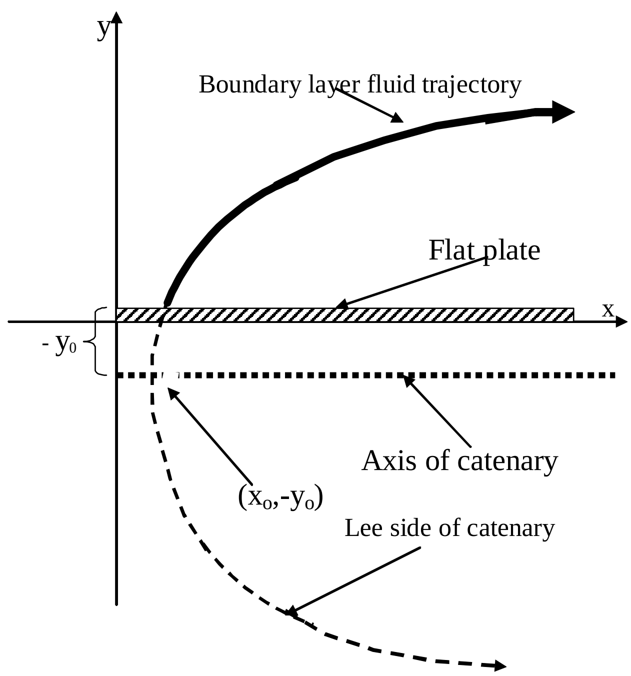

Figure 2.2.3(a): Negative y_o for a completely laminar flow

But we can also have a flow that is transitional between a lamina flow and a turbulent one whereby $y_o = 0$ as in figure 2.2.3(b).

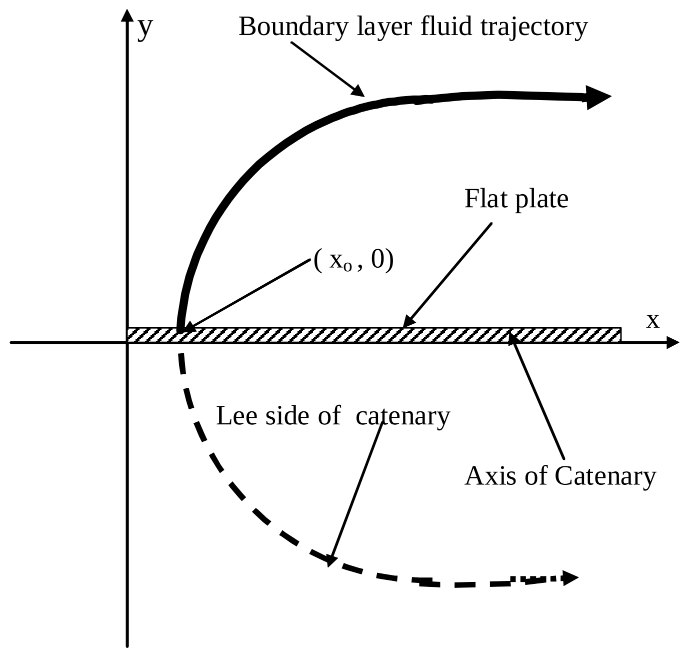

Figure 2.2.3(b): y_o=0 for a transitional flow

Yet thirdly we can have a completely turbulent flow in which y_o is positive as in figure 2.2.3(c).

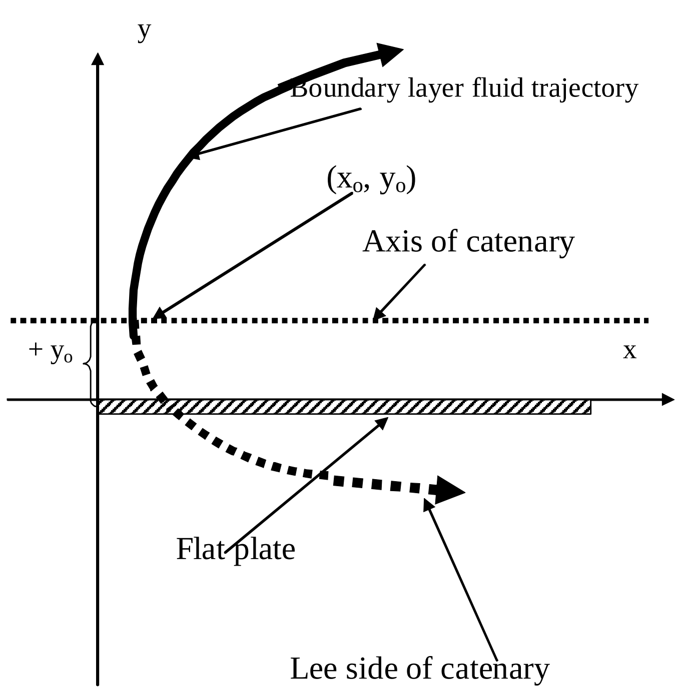

Figure 2.2.3(c): Positive y_o for a turbulent flow

Note that in the last case the boundary layer fluid does not make an contact with the flat plate surface.

We can similarly integrate equation (1.2.5) on the intervals [u_i, u_o] and [$\frac{\partial u_i}{\partial x_j}$, 0] to get

$$\frac{1}{2}\left(u_o^2 - u_i^2\right) - \upsilon \frac{\partial u_i}{\partial x_i} + \frac{1}{\rho}\left(P_{oi} - P_i\right) = 0 \qquad (2.2.4)$$

and on the intervals [u_j, 0] and [$\frac{\partial u_j}{\partial x_j}$, 0] to get

$$-\frac{1}{2}u_j^2 - \upsilon \frac{\partial u_j}{\partial x_j} + \frac{1}{\rho}\left(P_{oj} - P_j\right) = 0 \qquad (2.2.5)$$

Let us stop here with the notion that equations (2.2.4) and (2.2.5) are statements of Bernoulli's theorem for viscous fluids flowing in the respective mutually orthogonal directions. If we add them and invoke kinetic energy conservation we get

$$\upsilon \sum_i \frac{\partial u_i}{\partial x_i} = \frac{1}{\rho}\left(P_o - P\right) \qquad (2.2.6)$$

which is a relationship between pressure and continuity, the existence of one being a necessary condition for the other.

But we can also add equations (2.2.4) and (2.2.5) under continuity to get

$$P_o + \frac{1}{2}\rho u_0^2 = P + \frac{1}{2}\rho \sum_j u_j^2 \qquad (2.2.7)$$

which is a statement of the Bernoulli theorem under the condition of constant gravitation.

3.1 An analytical preview on previous results

The solution presented above proposes boundary layer flow parameters with a wider physical perspective than previously known. The Reynolds number is now neither the constant $R_e = \frac{u_o l}{\upsilon}$ representing similar flows nor the variable version $R_x = \frac{u_o x}{\upsilon}$ used for the semi-infinite flat plate. Instead the number stands for every point (X,Y) inside a fluid of kinematic viscosity υ flowing with a uniform velocity u_o such that

$$X = \frac{u_o}{\upsilon} x \quad \text{and} \quad Y = \frac{u_o}{\upsilon} y \qquad (3.1.1)$$

where $(x.y)$ are the rectangular Cartesian coordinates of that point. This newly defined coordinate system applies to every function that represents any phenomenon within that fluid flow.

Another boundary layer flow parameter affected in that similar manner is the similarity variable η. This variable is conceived of the scaling argument that boundary layer growth with the downstream coordinate x is given by $\delta(x) \approx \sqrt{\frac{\upsilon x}{u_o}}$.

Using this result one obtains the dimensionless variable $\eta = \frac{y}{\delta(x)}$ expressed variously as $\eta = y\sqrt{\frac{u_o}{\upsilon x}}$ or $\eta = \frac{y}{x}\sqrt{R_x}$. But this same result can be obtained from the flow profile

$$\frac{u_o}{2\upsilon}(x - x_o) + 1 = \frac{e^{\frac{u_o}{2\upsilon}(y+y_o)} + e^{-\frac{u_o}{2\upsilon}(y+y_o)}}{2} \qquad (2.2.3b)$$

by taking the second order Taylor series approximation of the RHS whereby we get the result

$$\frac{u_o}{2\upsilon}(x - x_o) = \left(\frac{u_o}{2\upsilon}(y + y_o)\right)^2 \qquad (3.1.2)$$

By reducing both x_o and y_o to zero we get an expression for which $\eta=1$. So the similarity variable is actually the quadratic approximation to the boundary layer flow profile.

3.2 Viscous drag on the flat plate surface

Apparently it is the position of the point (x_o,y_o) relative to the surface of the flat plate that determines the size of the viscous drag force experienced by the flat plate. If we take an elementary volume of fluid in contact with the two sides of a flat plate of area A to be given by $dV = 2Ady$, then equation (1.2.4) gives the elementary inertial force acting on the plate as[8]

$$dF = -2A\frac{\partial}{\partial y}\left(\frac{1}{2}\rho u^2\right)dy$$

On integration with respect to u on the interval $[u, u_o]$ this latter equation gives the force

[8] The same result is obtained for $dF = -2A\mu\frac{\partial^2 u}{\partial y^2}dy$ integrated over $\left[\frac{\partial u}{\partial y}, 0\right]$ and $[y, \infty)$

$$F = 4A\rho u_0^2 \frac{e^{\frac{u_0}{\upsilon}(y+y_o)}}{\left(e^{\frac{u_0}{\upsilon}(y+y_o)}+1\right)^2} \qquad (3.2.1)$$

The total drag force exerted on the plate surface is obtained by taking y = 0 to give

$$F_o = 4A\rho u_0^2 \frac{e^{\frac{u_0}{\upsilon}y_o}}{\left(e^{\frac{u_0}{\upsilon}y_o}+1\right)^2} \qquad (3.2.2)$$

Equation (3.2.2) is the well known proportionality to the square of the free stream velocity as expected, and from the definition [2] $C_D = \frac{F}{\frac{1}{2}A\rho u_o^2}$ the corresponding drag coefficient

$$C_D = 8\frac{e^{\frac{u_0}{\upsilon}y_0}}{\left(e^{\frac{u_0}{\upsilon}y_0}+1\right)^2} \qquad (3.2.3)$$

is a normal distribution represented by the curve in figure 3.2.1 below.

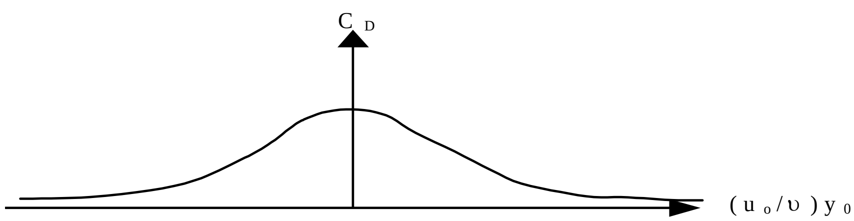

Figure 3.2.1: C_D from equation (3.2.3)

Few previuos empirical or experimental formulae have ever been close to equation (3.2.3). Stokes drag coefficient[3]

$$C_D = \frac{24}{R_e}$$

is known to work only for Reynolds numbers Re<0 whereas Carl Wilhem Oseen's [3] first approximation

$$C_D = \frac{24}{R_e}\left(1+\frac{3}{16}R_e\right)$$

is valid only for Re<1. Oseen's improved approximation

$$C_D = \frac{24}{R_e}\left(1+\frac{3}{16}R_e\right)^{\frac{1}{2}}$$

works no farther than Re=100 and no suitable approximate formulae has been suggested for the range of Reynolds numbers Re>1000. It has been stipulated [3] that over that range the drag coefficient for a flat plate no longer depends on the Reynolds number and values of C_D had to be plotted against the ratio of plate-length/plate-breadth instead to give the distribution in table 3.2.1.

Length/Breadth	1	2	4	10	18	∞
C_D	1.10	1.15	1.19	1.29	1.40	2.01

Table 3.2.1: Experimental values of C_D for Re>1000

On the other hand the accuracy of equation (3.2.3) extends further to its second order Taylor approximation

$$C_D = \frac{2}{\left(1+\frac{1}{8}\left(\frac{u_o}{\upsilon}y_o\right)^2\right)^2} \qquad (3.2.4(a))$$

which with the Reynolds number redefined to include $R_e = \frac{u_0}{\upsilon}y_0$ gives the equivalent alternative form

$$C_D = \frac{2}{\left(1+\frac{1}{8}R_e^2\right)^2} \qquad (3.2.4(b))$$

Equation (3.2.4(b)) gives the maximum value C_D = 2 at Re=0 as predicted by the experimental results in table 3.2.1 above.

Furthermore, equation (3.2.4(b)) is valid over the whole range $-\propto < R_e < +\propto$ and a plot of it in figure 3.2.2 compares to Duncan's [2] similar plot in figure 3.2.3. The difference between the two plots is only dimensional where Duncan uses $Log_{10} R_e = \frac{u_o c}{\upsilon}$ as the independent variable instead of the same form of the cordinate used in equation (3.2.4(a)). That is, instead of y_o Duncan uses c which he defines as the chord length of the flat plate.

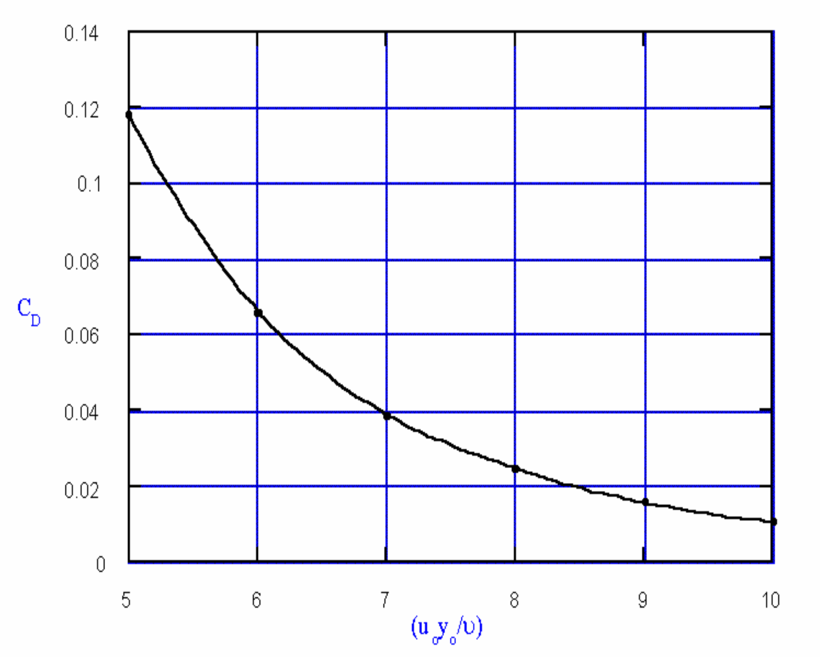

Figure 3.2.2: C_D from equation (3.2.4(a))

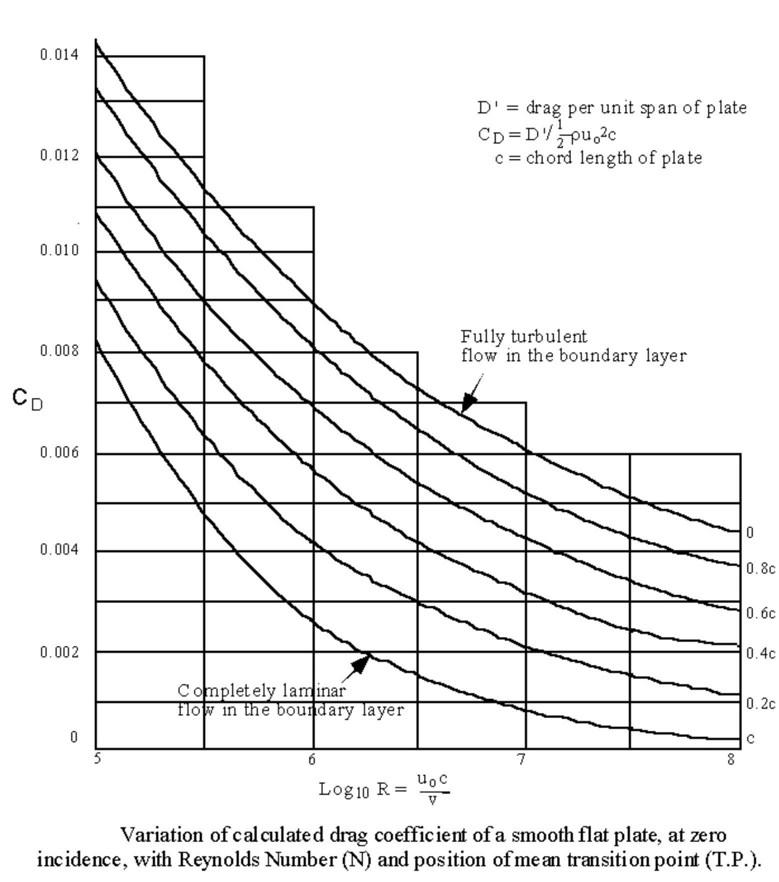

Figure 3.2.3: Duncan's Experimental C_D Curves [2]

3.3 Application to the uniform circular pipe

The point (x_o, y_o) similarly exists in the flow through a uniform circular pipe. The pipe then appears to consist of two cylinders, the inner real pipe of radius r_o surrounded by an outer pseudo-pipe of radius $R_o = r_o + y_o$ as shown in figure 3.3.1.

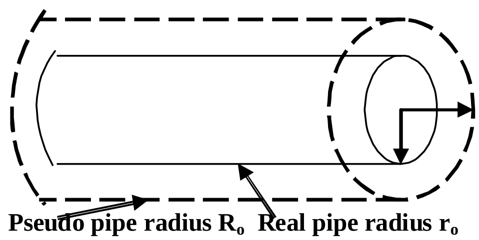

Figure 3.3.1: Pseudo pipe surrounding the real pipe

Assume that at steady flow the viscous drag force acting inside the pipe is constant so that

$$\mu \frac{d^2 u}{d r^2} = -C_0 \qquad (3.3.1)$$

A first indefinite integration of equation (3.3.1) with $\frac{du}{dr} = 0$ at r = 0 gives

$$\mu \frac{du}{dr} = -C_0 r$$

A second integration with u = 0 at r = R_o and u = u_o at r = 0 gives the velocity distribution

$$u = \frac{u_o}{R_o^2}\left(R_o^2 - r^2\right) \qquad (3.3.2)$$

that corresponds to the quadratic velocity profile in figure 3.3.2 below.

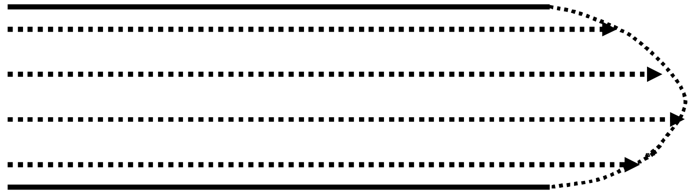

Figure 3.3.2: The quadratic velocity profile

For the velocity distribution in equation (3.3.2), the discharge of the pipe is given by the integral of

$$dQ = 2\pi\, r\, u\, d\, r$$

from $r=0$ to $r = r_o$ to give

$$Q = \frac{u_o \pi r_0^2}{2R_0^2}\left(2R_0^2 - r_0^2\right) \qquad (3.3.3)$$

With the ideal discharge

$$Q_0 = \pi r_o^2 u_o$$

and a small values of y_o compared to r_o, we get the discharge coefficient

$$C_d = \frac{1}{2} + \frac{r_o y_o}{\left(r_o + y_o\right)^2} \qquad (3.3.4)$$

This latter discharge always lies between 0.6% and 0.65% as expected from experiment [3], so it is practically not possible to get rid of the parameter y_o from the pipe. By assigning the proportionality $r_o = ky_o$ we deduce from equation (3.3.4) that the maximum discharge cannot exceed 0.75%, a result that has been confirmed by experiment.

From steady state and the input velocity being equal to the output velocity, a theoretical expectation arising from equation (3.3.3) suggests that the average flow velocity be given by

$$u_p = \frac{u_o}{2R_o^2}\left(2R_o^2 - r_o^2\right) \qquad (3.3.5)$$

However, the pressure at the points (x_o, y_o) is so high that it overrides the main flow at the pipe outlet to form venna contracta as shown in figure 3.3.3

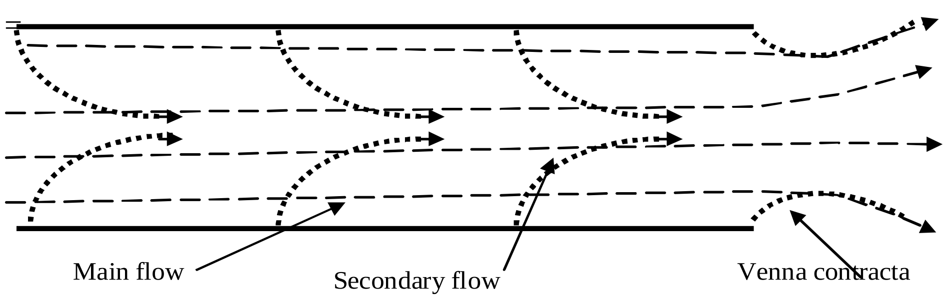

Figure 3.3.3: The secondary flow model velocity profile

so that in reality no quadratic velocity distribution actually exists.

Conclusion

The success of the energy formulation and solution of the boundary layer flow problem lies in the fact that very few previous authorities have been left out of it. Some conservative fluid mechanical analysts could call the complete abandon of the Navier-Stokes (N-S) equations a failure of the formulation, but ask them if they ever have solved the equations without truncating terms from them. And once terms have been removed in order to simplify the equations, every serious analyst cannot still refer to them as the Navier-Stokes equations, that way even Blasius ended up solving the Prandtl boundary layer flow problem rather than the N-S problem. So frankly speaking no exact solution for the Navier-Stokes equations exists, so the necessity for their replacement was long overdue.

But one may speak with certainty that the Hagen-Poisseuille formulation for the flow inside a uniform circular pipe has all along been a success. The theoretical solution of it is verifiable and has successfully been confirmed by experiment. And equation (1.2.4) is nothing but such a relationship only that this time it represents a boundary layer flow across a flat plate. Further still, equation (1.2.5) and Bernoulli's law are just one and the same. So unlike the Navier-Stokes formulation, equations arising from the energy formulation are much similar to previous ones that worked.

That boundary layer fluid mechanics has lacked a theoretical framework for the whole one hundred and fifty years of the existence of the N-S equations cannot be denied. No one ever solved the N-S equations without first invoking the Reynolds number Re, but then Re is a practical result established by Sir Osborne Reynolds. So the N-S equations have had to fall back onto some practical result for them to acquire a semblance to a valid fluid dynamical problem. In the energy formulation we have coordinates in the form of the same Reynolds numbers $\left(X = \frac{u_o}{\upsilon} x , Y = \frac{u_o}{\upsilon} y \right)$ but with their length parameter now properly and understandably defined. And the notion that the Reynolds number is part of every fluid flow coordinate is ample proof why Sir Osborne Reynolds found it in every measurement he took.

The foregoing view concerning the Reynolds number also applies to the concept of similarity, that is, even though fluid flows may exist in completely different dimensions of space coordinates (x,y), they shall be mathematically the same flow if they have the same value for the flow parameters $\left(\frac{u_o}{\upsilon} x , \frac{u_o}{\upsilon} y \right)$. There was an earlier confusion concerning similarity, the latter confusion which led to the dummy variable $\eta = y\sqrt{\frac{u_o}{\upsilon x}}$ being misnamed the similarity variable. But when plotted in the real plane the variable η corresponds to a whole quadratic distribution in x and y, so one actually wonders what a solution that is in terms of η actually represents in real space. The energy formulation gets η as a second order Taylor series approximation of the flow profile given in equation (2.2.3(b)) so that η is in fact the quadratic approximation to the boundary layer flow profile.

The solution to the energy formulation above works even better when it comes to its application to practical situations. Previously the velocity-squared force law was purely empirical and experimental, but now equation (3.2.2) represents the same law derived from first principles using equation (1.2.4). From equation (3.2.2) one also easily gets equation (3.2.3) which is an exact expression for the drag coefficient for the flat plate.. And unlike previous approximations which worked only on some limited range of Reynolds numbers, the second order Taylor approximation for equation (3.2.3) provides an expression that is valid for all Reynolds numbers.

The energy formulation has also been successfully substituted for the Hagen-Poiseuille formulation of the flow inside the uniform circular pipe and the results reflect on all the practically observed aspects of fluid transport in the pipes. Accordingly therefore, rather than have practical work verifying our theoretical findings; we have come up with a theoretical finding that has already been verified by experimental work in nearly all real life applications.

Apparently the most obvious reason for the failure of the N-S formulation is the inappropriateness of the momentum operator

$$\left\{\frac{\partial}{\partial \mathrm{t}} + (\vec{V} \bullet \vec{\nabla})\right\}$$

This operator has been derived from Newton's second law that defines a force as

$$\vec{F_i} = m\frac{d\vec{V}}{dt}$$

but this latter implies a total change in velocity with respect to time without any changes in space. So the action of converting this total derivative into a partial one with respect to both space and time is a purely mathematical one and has no physical connection. In fact, between $\vec{F}_1$ and $\vec{F}_2$ the corresponding expressions for the work done

$$\vec{F}_1 \bullet dr = m\left\{\frac{\partial \vec{V}}{\partial \mathrm{t}} + (\vec{V} \bullet \vec{\nabla})\vec{V}\right\} \bullet dr$$

and

$$\vec{F_2} \bullet dr = \left\{m\frac{\partial \overline{V}}{\partial \mathrm{t}} + \frac{1}{2}m\overrightarrow{\nabla}(\overrightarrow{V} \bullet \overrightarrow{V})\right\} \bullet dr$$

show every possibility that the former does not come out in accordance with the standard definition of work because it introduces foreign terms that do not fit into that definition..

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